STRONG PROJECTIVE WITNESSES

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Abstract. We show Shelah's original creature forcing from 1984 strongly preserves tight mad families. In particular, answering questions of Fischer and Friedman and Friedman and Zdomskyy, we show the constellation $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$ is consistent with the existence of a Δ^1_3 wellorder of the reals and tight mad families of sizes $\aleph_1, \aleph_2$ which are Π^1_1, Π^1_2-definable, respectively. Each of these projective definitions is of minimal possible complexity.

1. Introduction

In the following we reveal a combinatorial property of a rather well-known proper forcing notion $\mathbb{Q}$ introduced by Shelah in 1984 (see Definition 2.12, and [She84, Definition 6.8] for the original definition), to obtain the consistency of $\mathfrak{b} < \mathfrak{s}$ and thus establishing the independence of $\mathfrak{b}$ and $\mathfrak{s}$. The notion $\mathbb{Q}$ is the first instance of so-called creature forcings, which now encompass a broad class of posets (see [RS99]). Namely, we show in particular that $\mathbb{Q}$ satisfies an iterable preservation property (Definition 2.2) introduced in [GHT20], which guarantees the preservation of tight mad families under countable support iterations.

Theorem (Theorem 2.22)**.** Let $\mathbb{Q}$ be the forcing notion of Definition 2.12, and let $A \in V$ be a tight mad family. Let G be $\mathbb{Q}$-generic over V. Then ($\mathcal{A}$ is a tight mad family)$^{V[G]}$.

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The poset $\mathbb{Q}$ adds a generic real unsplit by the ground model reals in a similar manner as Mathias forcing, though unlike Mathias forcing, it is almost ω^ω-bounding and so countable support iterations of $\mathbb{Q}$ over a model of CH preserve the ground model reals unbounded, thus leading to the consistency of $\aleph_1 = \mathfrak{b} < \mathfrak{s} = \aleph_2$ (see [She84, Theorem 3.1]). Since $\mathfrak{b} \leq \mathfrak{a}$, the results of the current paper give an alternative proof and a natural strengthening of the above theorem. This newly discovered preservation property of Shelah's creature poset $\mathbb{Q}$ plays a crucial role in obtaining the following interesting result, lying at the intersection of descriptive set theory and set theory of the reals. Using a countable support iteration of S-proper forcing notions and the preservation of *nicely definable witnesses* to $\mathfrak{a}$ in such iterations we obtain our main result:

Theorem (Theorem 5.1)**.** It is consistent with $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \mathfrak{c} = \aleph_2$ that there exists a Δ^1_3 wellorder of the reals, a coanalytic tight mad family of size $\aleph_1$, and a Π^1_2 tight mad family of size $\aleph_2$.

Classical theorems of Mansfield, Mathias [Mat77], and Mansfield-Solovay, respectively, imply that these projective definitions are of minimal descriptive complexity, thus optimal. The above Theorem will be proved through a series of intermediary steps, beginning with the construction of a Δ^1_3 wellorder of the reals of Fischer and Friedman from [FF10], and establishing the following:

Theorem (Theorem 3.1)**.** It is consistent with $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \mathfrak{c} = \aleph_2$ that there exists a Δ^1_3-definable wellorder of the reals and moreover $\mathfrak{a} = \aleph_1$ is witnessed by a Π^1_1 tight mad family.

In the recent literature there is an increased interest in the projective definability of witnesses to cardinal characteristics; the relevant body of work includes, for example, [BFB22], [FFZ11], [FS21], [FFK13], [BK13], [FFST25] or [FSS25]. We extend this line of inquiry by investigating the projective definability of witnesses for all cardinals κ belonging to the set $\operatorname{spec}(\mathfrak{a}) = \{|\mathcal{A}| \mid \mathcal{A} \subseteq [\omega]^\omega, \mathcal{A} \text{ is mad}\}$. Towards this end we examine the S-proper countable support iteration of Friedman and Zdomskyy [FZ10] used in establishing the consistency of $\mathfrak{b} = \mathfrak{c} = \aleph_2$ with a Π^1_2-definable tight mad family, which is necessarily of cardinality $\mathfrak{c} = \aleph_2$. In particular, answering [FZ10, Question 18], we obtain:

Theorem (Theorem 4.1)**.** It is consistent that $\mathfrak{a} = \aleph_1 < \mathfrak{c} = \aleph_2$, and there exists a Π^1_1-definable tight mad family of size $\aleph_1$, and a Π^1_2-definable tight mad family of size $\aleph_2$.

The paper is organized as follows: In Section 2 we introduce the forcing notion $\mathbb{Q}$, the notion of a tight mad family and strong preservation of tightness, and prove Theorem 2.22. In Section 3 we give an account of the countable support iteration of Fischer and Friedman for adjoining a Δ^1_3 wellorder of the reals, and prove Theorem 3.1. Section 4 investigates Friedman-Zdomskyy poset mentioned above and proves Theorem 4.1. Building on the thus established results, in Section 5 we give our main result Theorem 5.1. The last section includes final remarks and open questions.

2. Creature Forcing

Define the relation $\leq^*$ of *eventual domination* on the collection ω^ω by letting $f \leq^* g$ if and only if there exists $n \in \omega$ such that for all $m \geq n$, $f(m) \leq g(m)$, and the relation *splits* on the collection $[\omega]^\omega$ by letting a split b if and only if both $b \cap a$ and $b \setminus a$ are infinite. In 1984 Shelah answered a question of Nyikos by showing that it is consistent that the minimal size of a family $\mathcal{F} \subseteq \omega^\omega$ which is unbounded with respect to $\leq^*$ can be strictly less than the minimal size of a splitting family $\mathcal{S} \subseteq [\omega]^\omega$, that is, a family $\mathcal{S}$ with the property that for any infinite $b \subseteq \omega$ there is $a \in \mathcal{S}$ which splits b. We denote with $\mathfrak{b}$ and $\mathfrak{s}$ these respective cardinalities. That $\mathfrak{s} < \mathfrak{b}$ is consistent was already shown in 1980 by Balcar and Simon [BPS80, Remark 4.7]; the inequality also holds in the Hechler model (see [She84, Section 4]). A family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* if $a \cap b$ is finite for every $a, b \in \mathcal{A}$, and such a familly is *maximal almost disjoint (mad)* if $\mathcal{A}$ is maximal with respect to inclusion among all almost disjoint families; equivalently, for every $b \in [\omega]^\omega$ there exists $a \in \mathcal{A}$ such that $a \cap b$ is infinite. We will always consider infinite mad families. The cardinal $\mathfrak{a}$ denotes the minimal size of a mad family; the cardinals $\mathfrak{a}$ and $\mathfrak{d}$ are independent.

One approach to obtaining a model in which $\mathfrak{s} = \aleph_2$ is with a countable support iteration of Mathias forcing $\mathbb{M}$, which adds a generic real $a \in [\omega]^\omega$ that is not split by any subset of the ground model. However Mathias forcing cannot be used to show the consistency of $\mathfrak{b} < \mathfrak{s}$, since the generic real a also has the property that its enumeration function $e_a \colon \omega \to a$ is a dominating real over V, thereby witnessing $\omega^\omega \cap V$ is no longer $\leq^*$-unbounded in the $\mathbb{M}$-generic extension. In other words, the Mathias reals grow too fast to preserve the unboundedness of $\omega^\omega \cap V$. To remedy the problem Shelah's original "creature forcing" $\mathbb{Q}$ (Definition 2.12), slows the growth of the generic by using a countable sequence of *finite logarithmic measures* (see Definiton 2.8) on the second coordinate of a Mathias condition, allowing for careful selection of the integers with which one extends a given finite approximation so to not dominate $\omega^\omega \cap V$. The $\mathbb{Q}$-generic reals are not split by any ground model family for similar reasons as with $\mathbb{M}$. The fact $\mathfrak{b} = \aleph_1$ in a countable support iteration of $\mathbb{Q}$ over a model of CH was established in [She84] via the notion of *almost ω^ω-boundedness*, see for example [Abr10] for further discussion. Shelah shows $\mathfrak{a} = \aleph_1$ also holds in the above mentioned extension by directly constructing a mad family in the ground model and proving the indestructibility of this specific mad family under $\mathbb{Q}$ and its iterations. In 1995 Alan Dow [Dow95] constructs a mad family in a model of $\mathfrak{p} = \mathfrak{c}$ with similar properties and shows that it is indestructible by countable support iterations of Miller forcing, establishing $\mathfrak{a} = \aleph_1$ in the Miller model. It is interesting to observe that both Shelah's and Dow's indestructible mad families discussed here are in fact tight, as shown in Propositions 2.6 and 2.7, respectively.

Since 1984 the theory of forcing preservation has considerably developed; see, for example, [She17] or [Gol93]. Likewise has been developed the theory of *indestructibility* of various witnesses to cardinal characteristics, in particular that of mad families; see [BY05], [Hru01], or [HGF03]. In 2020, Guzman, Hrušák, and Tellez study the preservation of tight mad families, where:

Definition 2.1. An almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ is *tight* if for all countable collections $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})^+$, there exists a single $a \in \mathcal{A}$ such that $a \cap b$ is infinite for each $b \in \mathcal{B}$.

Above, $\mathcal{I}(\mathcal{A})^+$ denotes the complements of the sets belonging to the ideal generated by $\mathcal{A}$ and the finite subsets of ω, $\mathcal{I}(\mathcal{A}) = \{b \subseteq \omega \mid \exists F \in [\mathcal{A}]^{<\omega}\ (b \subseteq^* \bigcup F)\}$.

Notice that if an almost disjoint family is tight, then it is immediately maximal. This strengthening initially appeared in the work of Malykhin [Mal89] in 1989, under the name of ω-mad families, and its connection to Cohen-indestructibility was established in Kurilić in 2001 [Kur01]. The existence of tight mad families follows from $\mathfrak{b} = \mathfrak{c}$ and a certain parametrized $\Diamond$-principle (see [HGF03]). Nonetheless it remains a long standing open question if ZFC proves the existence of tight mad families. Recently the following preservation property was introduced:

Definition 2.2 ([GHT20, Definition 7.1])**.** Let $\mathcal{A}$ be a tight mad family. We say a proper forcing notion $\mathbb{P}$ *strongly preserves the tightness of* $\mathcal{A}$ if for every $p \in \mathbb{P}$ and every countable elementary $\mathcal{M} \prec H_\theta$, where θ is a sufficiently large regular cardinal so that $\mathbb{P}, \mathcal{A}, p \in \mathcal{M}$, and for every $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap Y$ is infinite for all $Y \in \mathcal{I}(\mathcal{A})^+ \cap \mathcal{M}$, there exists $q \leq p$ such that q is $(\mathcal{M}, \mathbb{P})$-generic and $q \Vdash$ "$\forall \dot{Z} \in (\mathcal{I}(\mathcal{A})^+ \cap M[\dot{G}])(|\dot{Z} \cap B| = \omega)$". Such a q is called an $(\mathcal{M}, \mathbb{P}, \mathcal{A}, B)$*-generic condition.*

It is easy to see that if $\mathbb{P}$ is proper and preserves the tightness of some $\mathcal{A}$, then $\mathcal{A}$ remains tight in any $\mathbb{P}$-generic extension. Crucially:

Lemma 2.3 ([GHT20, Corollary 6.5])**.** Let γ be an ordinal, and let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ be a countable support iteration such that for all $\alpha < \gamma$, $\Vdash_{\mathbb{P}_\alpha}$" $\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\check{\mathcal{A}}$". Then $\mathbb{P}$ strongly preserves the tightness of $\mathcal{A}$.

The proof of the above follows the lines of the preservation of properness under countable support iterations (see [Abr10]; [She82]).

Examples of posets strongly preserving tightness are Miller forcing, Miller partition forcing, and Sacks forcing. Our Theorem 2.22 adds Shelah's original creature forcing to this list, thus providing an alternative proof of [She84, Theorem 3.1] (see Theorem 2.25).

An already recurring approach in establishing that a proper forcing strongly preserves tightness as in Definition 2.2 is to appeal to an *outer hull argument*.

Definition 2.4. If $\mathbb{P}$ is a forcing notion, $p \in \mathbb{P}$ and $\dot{Z}$ is a $\mathbb{P}$-name for a subset of ω, the *outer hull of* $\dot{Z}$ *with respect to* p is the set $W_p := \{m \in \omega \mid \exists r \leq p\ (r \Vdash m \in \dot{Z})\}$.

Fact 2.5 ([GHT20, Lemma 6.2])**.** For an almost disjoint family $\mathcal{A}$ and $\mathbb{P}$ a forcing notion, if $\dot{Z}$ is a $\mathbb{P}$-name for an element of $\mathcal{I}(\mathcal{A})^+$, then for any $p \in \mathbb{P}$, $W_p \in \mathcal{I}(\mathcal{A})^+$.

The above follows from noting that $\dot{Z}$ will be forced by p to be a subset of W_p. Our main strategy for establishing strong preservation of tightness will be to consider a *refined notion* of the outer hull and prove an analogue of Fact 2.5 for these refinements, see Claim 2.23 and Lemma 4.8.

We proceed with Shelah's and Dow's constructions of tight mad families mentioned earlier. The following can be found in the proof of [She84, Theorem 3.1] or [She17, Theorem 7.1, Chapter VI].

Proposition 2.6. (CH) There exists a tight mad family.

Proof. Fix an enumeration $\{\langle B_n^\alpha : n \in \omega\rangle \mid \alpha < \omega_1\}$ of sequences $\langle B_n^\alpha : n \in \omega\rangle$, with $B_n^\alpha \in [\omega]^{<\omega} \setminus \{\emptyset\}$ and $B_n^\alpha \cap B_m^\alpha = \emptyset$ for all distinct $m, n \in \omega$. By induction on $\alpha < \omega_1$, recursively define a family $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ as follows. First, choose a partition $\{A_n \mid n \in \omega\}$ of ω into infinite sets. For $\alpha \in [\omega, \omega_1)$, choose $A_\alpha \subseteq \omega$ so that A_α is almost disjoint from A_β for each $\beta < \alpha$ and for each $\beta < \alpha$, <u>if</u> for all $k \in \omega$ and $\alpha_j < \alpha$ for $j < k$, for all $m \in \omega$ the set $\{n \in \omega \mid \min(B_n^\beta) > m \wedge\ B_n^\beta \cap \bigcup_{j<k} A_{\alpha_j} = \emptyset\}$ is infinite, <u>then:</u>

(1) there exist infinitely many $n \in \omega$ such that $B_n^\beta \subseteq A_\alpha$, and

(2) for all $k \in \omega$ and $\alpha_j \leq \alpha$ for $j < k$, the set $\{n \in \omega \mid B_n^\beta \cap (\bigcup_{j<k} A_{\alpha_j}) = \emptyset\}$ is infinite.

Let $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$. It is straightforward to check that $\mathcal{A}$ is a tight mad family. $\square$

Proposition 2.7 ([Dow95, Lemma 2.3])**.** Let $\mathfrak{p} = \mathfrak{c}$. Then, there is a tight mad family $\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{c}\}$ such that for any fixed enumeration $\{\langle B_n^\alpha : n \in \omega\rangle \mid \alpha < \mathfrak{c}\}$ of $[[\omega]^{<\omega}]^\omega$, for any $\alpha < \mathfrak{c}$, <u>if</u> there exists $\beta < \alpha$ such that

$(*)$: for all $I \in \mathcal{I}(\bigcup_{\beta<\alpha} A_\beta)$, $B_n^\beta \cap I = \emptyset$ for infinitely many $n \in \omega$,

<u>then</u> there are infinitely many n such that $B_n^\beta \subseteq A_\alpha$.

Proof. We define $\mathcal{A}$ inductively by first letting $\{A_n \mid n \in \omega\}$ be any partition of ω into infinite sets. Suppose $\{A_\beta\}_{\beta<\alpha}$ have been constructed and let $\mathcal{I}_\alpha$ denote the ideal $\mathcal{I}(\bigcup_{\beta<\alpha} A_\beta)$. We can assume without loss of generality that $(*)$ holds for all $\beta < \alpha$. Next, for each finite partial function $s\colon \omega \to \alpha$, define F_s to be the set of all $x \in [\omega]^{<\omega}$ such that $x \cap \bigcup_{i\in\mathrm{dom}(s)} A_{s(i)} = \emptyset$, and $\forall i \in \mathrm{dom}(s)\ \exists m > \max\mathrm{dom}(s)\ (B_m^{s(i)} \subseteq x)$. Then $\mathcal{F} = \{F_s \mid s\colon \omega \to \alpha$, s is a finite partial function$\} \subseteq [[\omega]^{<\omega}]^\omega$ has the SFIP and is of cardinality $|\alpha^{<\omega}| < \mathfrak{c} = \mathfrak{p}$, so there exists an infinite $Y \subseteq [\omega]^{<\omega}$ such that $Y \subseteq^* F$ for each $F \in \mathcal{F}$. Let $A_\alpha = \bigcup Y$. To see A_α is as desired, let $\beta < \alpha$, and let $s\colon \omega \to \alpha$ be such that $s(0) = \beta$. Then $A_\beta \cap A_\alpha$ is contained in $Y \setminus F_s$, so as the latter is finite, so is the former.

Moreover the set $\{n \in \omega \mid B_n^\beta \subseteq A_\alpha\}$ is infinite. Note that as $Y \cap F_s$ is infinite, it is in particular nonempty, and if $y \in Y \cap F_s$ then there is $m > \max\mathrm{dom}(s)$ with $B_m^{s(0)} = B_m^\beta \subseteq y \subseteq A_\alpha$. Because $\max\mathrm{dom}(s)$ can be taken to be any $n \in \omega$, as we only require $s(0) = \beta$, the integer m above can be taken arbitrarily large. It is straightforward to check that $\mathcal{A}$ is a tight mad family. $\square$

Next we show that $\mathbb{Q}$ strongly preserves tightness. Our presentation of the poset follows Abraham's [Abr10] (see also [Fis08]).

Definition 2.8. For s a subset of ω, a *logarithmic measure on s* is a function $h\colon [s]^{<\omega} \to \omega$ such that for all $A, B \in [s]^{<\omega}$ and $\ell \in \omega$, if $h(A \cup B) \geq \ell + 1$, then either $h(A) \geq \ell$ or $h(B) \geq \ell$. A *finite logarithmic measure* is a pair (s, h) such that $s \subseteq \omega$ is finite and h is a logarithmic measure on s.

Lemma 2.9 ([She84]; [Fis08, Lemma 2.1.3])**.** If h is a logarithmic measure on s and $h(A_0 \cup \ldots \cup A_{n-1}) > \ell$, then there exists $j < n$ such that $h(A_j) \geq \ell - j$.

The *level* of a finite logarithmic measure (s, h) is the value $h(s)$ and is denoted level(h). If $A \subset s$ such that $h(A) > 0$, then A is called *h-positive.*

Definition 2.10. Let $P \subseteq [\omega]^{<\omega}$ be an upwards closed collection. The logarithmic measure h on $[\omega]^{<\omega}$ *induced by P* is defined inductively on the cardinality of $s \in [\omega]^{<\omega}$ as follows:

(1) $h(e) \geq 0$ for all $e \in [\omega]^{<\omega}$;
(2) $h(e) > 0$ if and only if $e \in P$;
(3) For all $\ell \geq 1$, $h(e) \geq \ell + 1$ if and only if $|e| > 1$ and for all $e_0, e_1 \subseteq e$ such that $e = e_0 \cup e_1$, then $h(e_0) \geq \ell$ or $h(e_1) \geq \ell$.

Then $h(e) = \ell$ if and only if $\ell \in \omega$ is maximal such that $h(e) \geq \ell$.

If h is as above and e is such that $h(e) \geq \ell$, then $h(a) \geq \ell$ for all sets $a \supseteq e$. In the following we will always assume an induced logarithmic measure is *non-atomic*, meaning there are no h-positive singletons. This assumption is necessary for the proof of the next lemma, which gives a sufficient condition for an induced logarithmic measure to take arbitrarily high values, and so allowing for the construction of pure extensions with desired properties. The following can be shown by a König's lemma argument.

Lemma 2.11 ([Abr10, Lemma 4.7], [Fis08, Lemma 2.1.9])**.** Let $P \subseteq [\omega]^{<\omega}$ be an upwards closed collection of nonempty sets, and let h be the induced logarithmic measure. Suppose that:

(†) for every $n \in \omega$ and every partition $\omega = A_0 \cup \ldots \cup A_{n-1}$, there is $i < n$ such that $[A_i]^{<\omega}$ contains some $x \in P$.

Then for every $n, k \in \omega$, and finite partition of ω into sets $A_0 \cup \ldots \cup A_{n-1}$, there exists $i < n$ and $x \subseteq A_i$ such that $h(x) \geq k$.

We now define our main forcing of interest.

Definition 2.12 ([She84, Definition 2.8], [Fis08, Definition 1.3.5])**.** Let $\mathbb{Q}$ be the partial order consisting of pairs $p = (u, T)$ such that $u \subseteq \omega$ is finite and T is a sequence $T = \langle t_i : i \in \omega \rangle$, where for all $i \in \omega$, t_i is a pair $t_i = (s_i, h_i)$, where h_i is a finite logarithmic measure on s_i, and such that:

(1) $\max(u) < \min(s_0)$;
(2) $\max(s_i) < \min(s_{i+1})$ for all $i \in \omega$;
(3) $\langle h(s_i) : i \in \omega \rangle$ is unbounded and strictly increasing.

For T as above, let $\mathrm{int}(t_i) = s_i$, and $\mathrm{int}(T) = \bigcup_{i \in \omega} \mathrm{int}(t_i)$. When $e \subseteq \mathrm{int}(t_i)$ is such that $h_i(e) > 0$, we say that e is *t_i-positive.* For conditions $(u_0, T_0), (u_1, T_1) \in \mathbb{Q}$, where $T_j = \langle t_i^j : i \in \omega \rangle$, $t_i^j = (s_i^j, h_i^j)$ for $j < 2$, define $(u_1, T_1) \leq (u_0, T_0)$ if and only if:

(5) u_1 end-extends u_0 and $u_1 \setminus u_0 \subseteq \mathrm{int}(T_0)$;
(6) $\mathrm{int}(T_1) \subseteq \mathrm{int}(T_0)$ and there is a sequence $\langle B_i : i \in \omega \rangle$ of finite subsets of ω such that $\max(B_i) < \min(B_{i+1})$ and for each $i \in \omega$, $s_i^1 \subseteq \bigcup_{j \in B_i} s_j^0$;
(7) For all $i \in \omega$ and $e \subseteq s_i^1$, if $h_i^1(e) > 0$ then there exists $j \in B_i$ such that $h_j^0(e \cap s_j^0) > 0$.

In the case $u_1 = u_0$, call (u_1, T_1) a *pure extension* of (u_0, T_0).

If $(\emptyset, T)$ is a condition in $\mathbb{Q}$, we will identify $(\emptyset, T)$ and T, and write simply $T \in \mathbb{Q}$. For $T = \langle t_i : i \in \omega \rangle \in \mathbb{Q}$ and $k \in \omega$, let

$$i_T(k) = \min\{i \in \omega \mid k < \min(\mathrm{int}(t_i))\}$$

and let $T \setminus k = \langle t_i : i \geq i_T(k) \rangle$. Then $T \setminus k \in \mathbb{Q}$ and $T \setminus k \leq T$. Similarly if $u \subseteq \omega$ is finite, $T \setminus \max(u) \in \mathbb{Q}$. By a slight abuse of notation, we will understand by $(u, T \setminus u)$ to mean the condition $(u, T \setminus \max(u))$ in the case $\max(u) \geq \min(\mathrm{int}(t_0))$. That $\mathbb{Q}$ satisfies Axiom A (see [Abr10, Definition 2.3]) and hence is proper can be established by the following.

Definition 2.13. For $n \in \omega$ and $(u_0, T_0), (u_1, T_1) \in \mathbb{Q}$, let $\leq_0$ be the usual partial order on $\mathbb{Q}$. Let $T_j = \langle t^j_i : i \in \omega \rangle$ for $j < 2$. Define

(1) $(u_1, T_1) \leq_1 (u_0, T_0)$ iff $(u_1, T_1) \leq_0 (u_0, T_0)$ and $u_1 = u_0$;
(2) for $n \geq 1$ let $(u_1, T_1) \leq_{n+1} (u_0, T_0)$ iff $(u_1, T_1) \leq_1 (u_0, T_0)$ and $t^1_i = t^0_i$ for all $i < n$.

In particular, $(u_1, T_1) \leq_1 (u_0, T_0)$ if and only if (u_1, T_1) is a pure extension of (u_0, T_0).

Definition 2.14. Given a sequence $\langle p_i : i \in \omega \rangle \subseteq \mathbb{Q}$, $p_i = (u, T_i)$, $T_i = \langle t^i_j : j \in \omega \rangle$ such that $p_{i+1} \leq_{i+1} p_i$ for all $i \in \omega$, define the *fusion* of $\langle p_i : i \in \omega \rangle$ to be the condition $q := (u, \langle t_j : j \in \omega \rangle)$ such that $t_j := t^{j+1}_j$ for all $j \in \omega$.

If q is the fusion of $\langle p_i : i \in \omega \rangle$, then $q \leq_{i+1} p_i$ for all $i \in \omega$. The following notion is crucial both for proving that $\mathbb{Q}$ is proper and for our preservation result.

Definition 2.15. For $(u, T) \in \mathbb{Q}$, with $T = \langle t_i : i \in \omega \rangle$, and D an open dense subset of $\mathbb{Q}$, we say (u, T) is *preprocessed for D and $k \in \omega$* if for every $v \subseteq k$ such that v end-extends u, if $(v, \langle t_j : j \geq k \rangle)$ has a pure extension in D, then already $(v, \langle t_j : j \geq k \rangle) \in D$.

Note that if $(u, T) \in \mathbb{Q}$ is preprocessed for D and k, then any extension of (u, T) is also preprocessed for D and k.

Lemma 2.16 ([Fis08, Lemma 1.3.9])**.** For every open dense subset $D \subseteq \mathbb{Q}$, every $k \in \omega$, and every $p \in \mathbb{Q}$, there exists $q \in \mathbb{Q}$ such that $q \leq_{k+1} p$ and q is preprocessed for D and k.

As a consequence of the existence of fusion for $\mathbb{Q}$:

Lemma 2.17. For every open dense $D \subseteq \mathbb{Q}$ and every $p \in \mathbb{Q}$ there exists a pure extension $q \leq p$ such that q is preprocessed for D and every $k \in \omega$.

Let $\dot{C}$ be a $\mathbb{Q}$-name for a subset of ω, and let $j \in \omega$. Let $\dot{C}(j)$ denote the name for the j-th element of $\dot{C}$. We say a condition p *decides* $\dot{C}(j)$ if there exists $\ell \in \omega$ such that $p \Vdash \dot{C}(j) = \check{\ell}$. For such

$\dot{C}$ and $j \in \omega$, let $E_{C(j)} = \{p \in \mathbb{Q} \mid p \text{ decides } \dot{C}(j)\}$. When $p \in \mathbb{Q}$ forces that $\dot{C}$ is infinite, the set $E_{C(j)}$ is open dense below p in $\mathbb{Q}$.

Lemma 2.18. Let $T \in \mathbb{Q}$, let $\dot{C}$ be a $\mathbb{Q}$-name for an infinite subset of ω and let $n, j \in \omega$. Let $v \subseteq n$. Then there is $R = \langle r_i : i \in \omega \rangle \in \mathbb{Q}$ such that $R \leq T$ and for all $i \in \omega$ and r_i-positive $s \subseteq \operatorname{int}(r_i)$, there is $w \subseteq s$ such that $(v \cup w, R \setminus \max(s))$ decides $\dot{C}(j)$.

Proof. Let $T = \langle t_i : i \in \omega \rangle$ with $t_i = (s_i, h_i)$. By Lemma 2.17 we can suppose that T is preprocessed for $E_{C(j)}$ and every $k \in \omega$. Let $\mathcal{P}_v(C(j))$ denote those $x \in [\operatorname{int}(T)]^{<\omega}$ such that:

(1) For some $k \in \omega$, $x \cap \operatorname{int}(t_k)$ is t_k-positive;
(2) There exists $w \subseteq x$ such that $(v \cup w, \langle t_i : i > \max(x) \rangle)$ decides $\dot{C}(j)$.

Note that $\mathcal{P}_v(C(j))$ is upwards closed. Let $h\colon [\omega]^{<\omega} \to \omega$ be the logarithmic measure induced by $\mathcal{P}_v(C(j))$. We will show that h takes arbitrarily high values by establishing (†) of Lemma 2.11.

Fix $M \in \omega$ and a finite partition $\omega = A_0 \cup \ldots A_{M-1}$. First find $T' \leq T$ such that $\operatorname{int}(T') \subseteq A_N$ for some $N < M$; such an extension exists by König's lemma argument (see, for example, [Fis08]). Since $(v, T' \setminus v) \in \mathbb{Q}$, there is $w \subseteq \operatorname{int}(T' \setminus v)$ and R such that $(v \cup w, R)$ is a condition in $\mathbb{Q}$ extending $(v, T' \setminus v)$ and $(v \cup w, R)$ decides the value of $\dot{C}(j)$. Since $w \subseteq \operatorname{int}(T' \setminus v)$ is finite, by definition of the extension relation there are m_0, m_1 such that $w \subseteq \bigcup_{m \in [m_0, m_1]} \operatorname{int}(t'_m)$. Let $x := \bigcup_{m \in [m_0, m_1]} \operatorname{int}(t'_m)$ and note $x \in [A_N]^{<\omega}$. As $T' \setminus v \in \mathbb{Q}$ we may assume there is at least one $m \in [m_0, m_1]$ such that $\operatorname{int}(t'_m)$ is t'_m-positive. Then $T' \setminus v \leq T \setminus v$ and so there is $k \in \omega$ such that $h_k(\operatorname{int}(t'_m) \cap \operatorname{int}(t_k)) = h_k(x \cap \operatorname{int}(t_k)) > 0$. Therefore $x \subseteq \operatorname{int}(T') \subseteq A_N$ satisfies (1) in the definition of $\mathcal{P}_v(C(j))$.

We can also show that (2) holds. We have that $w \subseteq x$ is such that $(v \cup w, R) \in E_{C(j)}$, but T was preprocessed for $E_{C(j)}$ and $\max w$, and $(v \cup w, T \setminus w)$ has a pure extension into $E_{C(j)}$ so already $(v \cup w, T \setminus w) \in E_{C(j)}$. Altogether we have found $x \in \mathcal{P}_v(C(j)) \cap [A_N]^{<\omega}$, verifying (†).

We can now define $R = \langle r_n : n \in \omega \rangle$, where $r_n = (x_n, g_n)$, inductively as follows. Clearly $\mathcal{P}_v(C(j))$ is nonempty, as we just showed above, so pick $x_0 \in \mathcal{P}_v(C(j))$, and let $g_0 := h \restriction \mathcal{P}(x_0)$. Assuming $r_i = (x_i, g_i)$ defined for all $i \leq n$ so that $\max(x_i) < \min(x_{i+1})$ and $g_i(x_i) < g_{i+1}(x_{i+1})$ for $i < n$, since h takes arbitrarily high values there is $x_{n+1} \in \mathcal{P}_v(C(j))$ with $h(x_{n+1}) > h(x_n)$. We can assume $\max(x_n) < \min(x_{n+1})$, since otherwise h is bounded. Define $g_{n+1} := h \restriction \mathcal{P}(x_{n+1})$. This completes the definition of R.

Then $R \in \mathbb{Q}$, and it is routine to check R extends T. To see R is as desired, if $s \subseteq x_i$ is r_i-positive, by (2) of the definition of $\mathcal{P}_v(C(j))$ there is $w \subseteq s$ such that $(v \cup w, \langle t_i : i > \max(s) \rangle)$ decides $C(j)$, but $(v \cup w, \langle r_i : i > \max(s) \rangle) \leq (v \cup w, \langle t_i : i > \max(s) \rangle)$ and so makes the same decision. $\square$

Remark 2.19. If a condition R in $\mathbb{Q}$ has the property as in the conclusion of the above lemma, then any further extension retains this same property.

To obtain the next lemma consider each $v \in \mathcal{P}(n)$ and repeatedly apply Lemma 2.18 and Remark 2.19 (see e.g. [Fis08]).

Lemma 2.20. For any $T \in \mathbb{Q}$, $n, j \in \omega$ and $\dot{C}$ a $\mathbb{Q}$-name for an infinite subset of ω, there exists $R = \langle r_i : i \in \omega \rangle \in \mathbb{Q}$ such that $R \leq T$ and for all $v \subseteq n$, for all $i \in \omega$ and r_i-positive $s \subseteq \mathrm{int}(r_i)$, there is $w \subseteq s$ such that $(v \cup w, R \setminus s)$ decides $\dot{C}(j)$.

Corollary 2.21. For any $(u, T) \in \mathbb{Q}$, any $n, j \in \omega$, and any $\dot{C}$ a $\mathbb{Q}$-name for an infinite subset of ω, there exists $(u, R) \leq_{n+1} (u, T)$ such that for all $v \subseteq n$, for all $i \geq n$ and for every $s \subseteq \mathrm{int}(r_i)$ which is r_i-positive, there exists $w \subseteq s$ such that $(v \cup w, R \setminus s)$ decides the value of $\dot{C}(j)$.

The central result of this section is the following.

Theorem 2.22. *Let $\mathcal{A}$ be a tight mad family. For every $p \in \mathbb{Q}$ and $M \prec H_\theta$ countable elementary submodel, where θ is sufficiently large, containing $\mathbb{Q}, p, \mathcal{A}$, and every $B \in \mathcal{I}(\mathcal{A})$ such that $|B \cap Y| = \aleph_0$ for all $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$, and $\dot{Z}$ a $\mathbb{Q}$-name for an element of*

$\mathcal{I}(\mathcal{A})^+$ in M, there exists a pure extension $q \leq p$ such that q is $(M, \mathbb{Q}, \mathcal{A}, B)$-generic.

Proof. Fix θ, M and B as above, and let (u, T_0) be a condition in $\mathbb{Q} \cap M$. Let $\{D_n \mid n \in \omega\}$ enumerate all open dense subsets of $\mathbb{Q}$ in M, and let $\{\dot{Z}_n \mid n \in \omega\}$ enumerate all $\mathbb{Q}$-names for subsets of ω in M which are forced to be in $\mathcal{I}(\mathcal{A})^+$ such that each name appears infinitely often. We inductively define a sequence $\langle q_n : n \in \omega\rangle$ of conditions in $\mathbb{Q} \cap M$, where $q_0 = (u, T_0)$, $q_n = (u, T_n)$ with $T_n = \langle t_i^n : i \in \omega\rangle$, such that the following are satisfied:

(1) For all $n \in \omega$, $q_{n+1} \leq_{n+1} q_n$;
(2) q_{n+1} is preprocessed for D_n and every $k \in \omega$;
(3) For all $v \subseteq n$, for all $i \geq n$ and $s \subseteq \mathrm{int}(t_i^{n+1})$ which is t_i^{n+1}-positive, if v is an end-extension of u, then for some $w \subseteq s$, $((v \cup w), \langle t_j^{n+1} : j > \max(s)\rangle) \Vdash (\dot{Z}_n \cap B) \setminus n \neq \emptyset$.

Suppose q_n has been constructed. To define q_{n+1} consider the condition $(u, \langle t_i^n : i \geq n\rangle) \leq q_n$. By Lemma 2.17, there is a pure extension $(u, \langle t_i' : i \geq n\rangle) \leq (u, \langle t_i^n : i \geq n\rangle)$ which is preprocessed for D_n and every $k \in \omega$. Let $q_n^0 = (u, \langle t_i^{n,0} : i \in \omega\rangle)$, where for $i < n$, $t_i^{n,0} = t_i^n$, and $t_i^{n,0} = t_i'$ for $i \geq n$. Then $q_n^0 \leq_{n+1} q_n$ and q_n^0 satisfies (2). Next, consider the set

$$\begin{aligned} W_{n+1} = \{m \in \omega \mid \exists r = (u, \langle t_i' : i \in \omega\rangle) \leq_{n+1} q_n^0 \text{ satisfying:} \\ \forall v \subseteq n\ \forall i \geq n\ \forall s \subseteq \mathrm{int}(t_i')\ [s \text{ is } t_i'\text{-positive} \Rightarrow \\ \exists w \subseteq s(v \cup w, \langle t_i' : i > \max(s)\rangle) \Vdash m \in \dot{Z}_n]\} \end{aligned}$$

Claim 2.23. *$W_{n+1} \in \mathcal{I}(\mathcal{A})^+ \cap M$.*

Proof. We have that $W_{n+1} \in M$ since it is definable from the forcing relation and from $\mathbb{Q}$, q_n^0, and $\dot{Z}_n$, which are all assumed to be elements of M. To see $W_{n+1} \notin \mathcal{I}(\mathcal{A})$, let F be a finite subset of $\mathcal{A}$. We will show that $W_{n+1} \setminus \bigcup F$ is infinite. Since $\Vdash_{\mathbb{Q}} \dot{Z}_n \in \mathcal{I}(\mathcal{A})^+$, in particular q_n^0 forces that $\dot{Z}_n \setminus \bigcup F$ is infinite. Let $\dot{C}$ be a $\mathbb{Q}$-name for the set $\dot{Z}_n \setminus \bigcup F$.

By Corollary 2.21, for all $k \in \omega$ there is $q^j \leq_{n+1} q_n^0$, where $q^j = (u, R_j)$ and $R_j = \langle r_i^j : i \in \omega\rangle$ such that for all $v \subseteq n$, for all $i \geq n$ and r_i^j-positive $s \subseteq \mathrm{int}(r_i^j)$, there is $w \subseteq s$ such that $(v \cup w, R_j \setminus s)$ decides $\dot{C}(j)$ and $j > k$. So there exists $m_j \in \omega$ such that $(v \cup w, R_j \setminus s) \Vdash \dot{C}(j) = \check{m}_j$. Note that if $m_j = \dot{C}(j)$, then $m_j \geq j > k$. Therefore for all $k \in \omega$ there exists $m_j > k$ such that $m_j \in W_{n+1}$, witnessed by q^j as above, and moreover $m \notin \bigcup F$ since $q^j \Vdash \check{m}_j \notin \bigcup F$. Thus $W_{n+1} \setminus \bigcup F$ is infinite, and as $F \in [\mathcal{A}]^{<\omega}$ was arbitrary this proves the claim. $\square$

By assumption on the set B, there exists $m_{n+1} \in W_{n+1} \cap B$ such that $m > n$. Let $r = (u, \langle t'_i : i \in \omega \rangle) \leq_{n+1} q_n^0$ be given by $m_{n+1} \in W_{n+1}$, and define $q_{n+1} = (u, \langle t_i^{n+1} : i \in \omega \rangle)$ such that $t_i^{n+1} = t'_i$ for all $i \in \omega$. Since $r \leq_{n+1} q_n^0 \leq_{n+1} q_n$, we have that $q_{n+1} \leq_{n+1} q_n$. This completes the inductive construction.

Let $q = (u, T)$ be the fusion of the q_n's (see Definition 2.14), so $T = \langle t_i : i \in \omega \rangle$ with $t_i = t_i^{i+1}$ for all $i \in \omega$. We show q is $(M, \mathbb{Q})$-generic by showing that for all $n \in \omega$, the set $D_n \cap M$ is predense below q. Towards this end let $r = (v, R)$ be an arbitrary extension of q; as D_n is dense there exists $w \subseteq \mathrm{int}(R)$ such that $(v \cup w, R \setminus \max(w)) \in D_n$. Then as $(v \cup w, R \setminus \max(w)) \leq (u, T_{n+1} \setminus \max(w)) \leq q_{n+1}$ and q_{n+1} is preprocessed for D_n and $\max(w)$, already $r' = (v \cup w, T_{n+1} \setminus \max(w)) \in D_n$. Then $r' \in M$ and r, r' are compatible, as witnessed by the condition $(v \cup w, R \setminus \max(w))$.

Next, we show $q \Vdash |\dot{Z}_n \cap B| = \omega$ for every $n \in \omega$. Fix n, and let $(v, R) \leq q$ be arbitrary; it suffices to show that for every $k \in \omega$ there exists an extension of (v, R) which forces $(\dot{Z}_n \cap B) \setminus k \neq \emptyset$. Find $i \in \omega$ such that $i > k$, $v \subseteq i$, $\dot{Z}_n = \dot{Z}_i$ and $s = \mathrm{int}(R) \cap \mathrm{int}(t_i)$ is t_i-positive. The fact that s is $t_i = t_i^{i+1}$-positive and $r \leq q \leq q_{i+1}$ implies, by item (3), that there exists $w \subseteq s$ such that

$$(v \cup w, \langle t_j^{i+1} : j > \max(s) \rangle) \Vdash m_{i+1} \in \dot{Z}_i,$$

where $m_{i+1} \in B$ and $m_{i+1} \geq i > k$. Then $(v \cup w, R \setminus s)$ is an extension both of (v, R) and of the condition $(v \cup w, \langle t_j^{i+1} : j > \max(s) \rangle)$, so by the latter, forces $m_{i+1} \in (B \cap \dot{Z}_n) \setminus k$. This completes the proof that q is an $(M, \mathbb{Q}, \mathcal{A}, B)$-generic condition. $\square$

The following is a straightforward density argument.

Lemma 2.24. Let G be $\mathbb{Q}$-generic over V, let $\mathcal{S} \subseteq [\omega]^\omega$ be an element of V, and let $a = \{s \subseteq \omega \mid \exists T\ (s, T) \in G\}$. Then for all $b \in \mathcal{S}$, $(b \text{ does not split } a)^{V[G]}$.

Theorem 2.25 ([She84, Theorem 3.1])**.** *(CH) Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a countable support iteration such that for all $\alpha < \omega_2$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the partial order $\mathbb{Q}$ of Definition 2.12. Let G be $\mathbb{P}_{\omega_2}$-generic over V. Then $V[G] \models \aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$.*

Proof. Since $V \models \mathsf{CH}$ we can fix a tight mad family $\mathcal{A} \in V$. Let G be $\mathbb{P}_{\omega_2}$-generic over V. For all $\alpha < \omega_2$, by Theorem 2.22, $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is a proper forcing which strongly preserves the tightness of $\mathcal{A}$, so using Lemma 2.3, $\mathbb{P}_{\omega_2}$ is proper and preserves the tightness and hence maximality of $\mathcal{A}$. Therefore $\mathcal{A}$ witnesses $(\mathfrak{a} = \aleph_1)^{V[G]}$. Let $\mathcal{S} \subseteq [\omega]^\omega$ be family of cardinality $< \aleph_2$. Since $\mathbb{P}_{\omega_2}$ is $\aleph_2$-cc (see [Abr10, Theorem 2.10]), there exists $\alpha < \omega_2$ such that $\mathcal{S} \in V[G_\alpha]$, where $G_\alpha = G \cap \mathbb{P}_\alpha$ is $\mathbb{P}_\alpha$-generic over V. By definition of $\dot{\mathbb{Q}}_\alpha$ and Lemma 2.24, in $V[G_{\alpha+1}]$, $\mathcal{S}$ is not a splitting family, so also this holds in $V[G]$. Therefore $(\mathfrak{s} = \aleph_2)^{V[G]}$. □

3. Sacks coding and tightness

Recall that under $V = L$ there exists a Δ^1_2 wellorder of the reals [Göd39]; this complexity is optimal by the Lebesgue measurability of analytic sets. Conversely, Mansfield's theorem states that if there exists a Σ^1_2 wellordering of the reals, then all reals are constructible. Using a finite support iteration of ccc forcings, Harrington [Har77, Theorem B] showed that a $\boldsymbol{\Delta}^1_3$ wellordering is consistent with $\neg\mathsf{CH}$ and Martin's Axiom (MA); Friedman and Caicedo showed that the Bounded Proper Forcing Axiom (BFPA) and $\omega_1 = \omega_1^L$ imply the existence of a Σ^1_3 wellorder of the reals [CF11]. However in these last two constructions, the forcing axioms rendered all cardinal characteristics equal to $\mathfrak{c}$. The question of projective wellorderings of the reals in models with nontrivial structure of cardinal characteristics of the continuum was first addressed by Fischer and Friedman in

2010 [FF10]. Using a countable support iteration of S-proper forcing notions they showed a Δ^1_3 wellorder of the reals is compatible with $\mathfrak{c} = \aleph_2$ and each of the following inequalities: $\mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$, $\mathfrak{b} < \mathfrak{g}$. This was made possible by defining a new forcing notion, *Sacks coding*, which uses Sacks reals to code the wellorder and gives a way of forcing a Δ^1_3 wellorder with an ω^ω-bounding iteration.

Definable mad families of size $\mathfrak{c}$ can be obtained from a definable wellorder of $\mathbb{R}$, bringing into focus the study if the projective complexity of such families. Mathias [Mat77] was the first to do this, when in 1969 he showed no analytic almost disjoint family can be maximal. Subsequent work revealed further tension between mad families and determinacy assumptions; see, for example, [NN18], [BHST22], [Tö18], and references therein. On the other hand, in L there are Σ^1_2 mad families, as a recursive construction along the Σ^1_2-good wellorder $\leq_L$ of the constructible reals yields such a family. This was significantly improved by Miller [Mil89, Theorem 8.23], who obtained a coanalytic mad family under $V = L$.

It is interesting to ask whether one can obtain a cardinal preserving generic extension with a Δ^1_3 wellorder of the reals and $\neg\mathsf{CH}$, while simultaneously controlling values of cardinal characteristics as well as the definability of the witnesses to those characteristics. In 2022 Bergfalk, Fischer, and Switzer established that a Δ^1_3 wellorder of the reals is consistent with $\mathfrak{a} = \mathfrak{u} < \mathfrak{i}$, $\mathfrak{a} = \mathfrak{i} < \mathfrak{u}$, and $\mathfrak{a} < \mathfrak{u} = \mathfrak{i}$, with the added feature that the witnesses to the cardinal characteristics of value $\aleph_1$ can be taken to be coanalytic. In particular, they show that Sacks coding strongly preserves tightness of *nicely definable* tight mad families ([BFB22, Lemma 4.3]). Making use of this result and Theorem 2.22 we obtain:

Theorem 3.1. *It is consistent that* $\mathfrak{a} = \aleph_1 < \mathfrak{s} = \aleph_2$ *and that there exists a* Δ^1_3 *definable wellorder of the reals, and a* Π^1_1 *tight mad family of size* $\aleph_1$.

The above answers [FF10, Questions 7.(1)&(2)]. Our strategy in proving the above is to force with a countable support iteration over $V = L$ of S-proper forcings which strongly preserve the tightness of a coanalytic tight mad family $\mathcal{A} \in V$. Along the way we construct the wellorder $<_G = \bigcup_{\alpha<\omega_2} <_\alpha$ by defining the initial segments $<_\alpha$, where $<_\alpha$ wellorders the reals of $L^{\mathbb{P}_\alpha}$ using a canonical wellordering via $\leq_L$ of $\mathbb{P}_\alpha$-names for reals. Using appropriate bookeeping, at stage α we add a Sacks-generic real coding a pair of reals $x, y \in V^{\mathbb{P}_\alpha}$ such that $x <_\alpha y$. The way in which the α-th generic real codes these intial stages of the iteration provides the Δ^1_3-definability of the wellorder. More precisely at stage α, the Sacks-generic real r_α will code a sequence $\vec{C}_\alpha = \langle C_{\alpha+m} : m \in \Delta(x * y)\rangle$ of generic clubs in ω_1, as well as a set $Y \subset \omega_1$, where $\Delta(x * y) \subseteq \omega$ is a recursive coding of the pair (x, y), thus encoding a a pattern of stationary/nonstationary in apriori fixed sequence $\langle S_{\alpha+m} : m \in \omega\rangle \in L$, of stationary costationary subsets of ω_1^L. Note that, if r_α codes the pair (x, y), then $L[r_\alpha] \models \Delta(x * y) \subseteq \{m \in \omega \mid S_{\alpha+m} \text{ is nonstationary}\}$. If we can guarantee that for no $m \notin \Delta(x * y)$, $S_{\alpha+m}$ loses its stationarity in the final extension, then the stationarity-nonstationarity pattern encodes the wellorder $<_G$ as follows:

$$x <_G y \Leftrightarrow \exists \alpha < \omega_2 \; L[r_\alpha] \models \Delta(x * y) = \{m \in \omega \mid S_{\alpha+m} \text{ is nonstationary}\}$$

To make the latter a (lightface) projective definition of $<_G$, the set Y is added after $\vec{C}_\alpha$ in order to localize the generic nonstationarity of each $S_{\alpha+m}$ to a large class of countable transitive ZF^- models (Definition 3.7 below); this uses what is sometimes referred to as "David's Trick". Roughly, this allows us to bound the initial existential quantification to $\omega_2^{\mathcal{M}}$, where $\mathcal{M}$ belongs to the said class of countable transitive models, and $\mathcal{M}$ contains the real r_α. We proceed by introducing some key to our construction forcing notions.

3.1. **Club shooting.** Baumgartner, Harrington, and Kleinberg introduced in [BHK76] a cardinal preserving forcing notion which, given a stationary costationary $S \subseteq \omega_1$, adds a closed unbounded $C \subseteq \omega_1$ such that $C \cap S = \emptyset$. The forcing is often referred to as *club shooting.*

Definition 3.2. Let $S \subseteq \omega_1$ be a stationary costationary set. Define $Q(S)$ to be the partial order consisting of closed, bounded subsets of $\omega_1 \setminus S$, ordered by end-extension.

Lemma 3.3. The following hold:

(1) $Q(S)$ is ω-distributive, thus does not add new reals.
(2) Let G be $Q(S)$-generic. Then $C_G = \bigcup\{d \in Q(S) \mid d \in G\}$ is a club in $\omega_1^{V[G]}$ witnessing $(\check{S}$ is nonstationary$)^{V[G]}$.

Proof. See [Jec03, Chapter 25] or [Cum10, Section 6]. □

By item (2) above, $Q(S)$ is not a proper forcing notion, however, it still retains many of the desirable properties of a proper forcing.

Definition 3.4. Let $S \subseteq \omega_1$ be a stationary set. A forcing notion $\mathbb{P}$ is *S-proper* if for all countable elementary submodels $M \prec H_\theta$, with θ sufficiently large, and such that $M \cap \omega_1 \in S$, for every $p \in \mathbb{P} \cap M$ there is $q \leq p$ which is $(M, \mathbb{P})$-generic.

A proof of the following can be found in [Gol98, Theorem 3.7]

Lemma 3.5.

(1) If $S \subseteq \omega_1$ is stationary and $\mathbb{P}$ is S-proper, then $\mathbb{P}$ preserves ω_1 as well as the stationarity of any stationary subset of S.
(2) $Q(S)$ is $(\omega_1 \setminus S)$-proper.

Lemma 3.6.

(1) Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ be a countable support iteration such that for all $\alpha < \delta$, $\Vdash_{\mathbb{P}_\alpha}$"$\dot{\mathbb{Q}}_\alpha$ is an S-proper poset". Then $\mathbb{P}_\delta$ is S-proper. Moreover, if $\mathcal{A}$ is a tight mad family in the ground model and for all $\alpha < \omega_2$, $\Vdash_{\mathbb{P}_\alpha}$"$\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\check{\mathcal{A}}$", then also $\mathbb{P}_\delta$ strongly preserves the tightness of $\mathcal{A}$.
(2) Assume CH, and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ be a countable support iteration of S-proper posets of length $\delta \leq \omega_2$, such that for all $\alpha < \delta$, $\Vdash_{\mathbb{P}_\alpha}$"$|\dot{\mathbb{Q}}_\alpha| = \omega_1$". Then $\mathbb{P}_\delta$ is $\aleph_2$-cc. If also $\delta < \omega_2$, then CH holds in $V^{\mathbb{P}_\delta}$.

Proof. Easy modification of corresponding properness arguments. □

3.2. **Localization.** The forcing notion known as Localization has its roots in René David's work [Dav82] on absolute Π^1_2 singletons. Below we let ZF^- denote ZF without the Powerset Axiom.

Definition 3.7. A transitive model $\mathcal{M}$ of ZF^- is called *suitable* if $\omega_2^{\mathcal{M}}$ exists and $\omega_2^{\mathcal{M}} = \omega_2^{L^{\mathcal{M}}}$.

Throughout the rest of this section we assume V is a generic extension of L via a cofinality preserving forcing extension.

Definition 3.8. For $X \subseteq \omega_1$ and a Σ_1-sentence $\varphi(\omega_1, X)$ with parameters ω_1 and X such that φ holds in all suitable models $\mathcal{M}$ with $\omega_1, X \in \mathcal{M}$, denote by $\mathcal{L}(\varphi)$ the set of all functions $r\colon |r| \to 2$ where $|r| = \mathrm{dom}(r)$ is a countable limit ordinal, and such that:

(1) if $\gamma < |r|$ then $\gamma \in X$ if and only if $r(2\gamma) = 1$;
(2) if $\gamma \leq |r|$ and $\mathcal{M}$ is a countable suitable model such that $\gamma = \omega_1^{\mathcal{M}}$ and $r \restriction \gamma \in \mathcal{M}$, then $\mathcal{M} \models \varphi(\gamma, X \cap \gamma)$.

The extension relation is end-extension.

Each $r \in \mathcal{L}(\varphi)$ is an approximation to the characteristic function of a subset $Y \subseteq \omega_1$ such that $\mathrm{Even}(Y) = \{\gamma \mid 2\gamma \in Y\} = X$. The "odd part" of r, i.e. the values r takes on ordinals of the form $2\gamma+1$, is used for item (1) below.

Lemma 3.9. The following hold.

(1) ([FF10, Lemma 1]) For every $r \in \mathcal{L}(\varphi)$ and countable limit ordinal $\gamma > |r|$, there exists $r' \leq r$ such that $|r'| = \gamma$.
(2) ([FF10, Lemma 2]) If G is $\mathcal{L}(\varphi)$-generic and $\mathcal{M}$ is a countable suitable model such that $\bigcup G \restriction \omega_1^{\mathcal{M}} \in \mathcal{M}$, then $\mathcal{M} \models \varphi(\omega_1^{\mathcal{M}}, X \cap \omega_1^{\mathcal{M}})$.
(3) ([FF10, Lemma 3, Lemma 4]) $\mathcal{L}(\varphi)$ is proper, and moreover does not add new reals.

3.2.1. *Sacks Coding.* Sacks coding, or coding with perfect trees, was first defined by Fischer and Friedman [FF10]. Recall a tree $T \subseteq 2^{<\omega}$ is *perfect* if for all $s \in T$ there exists an extension t of s such that $t \in T$ and $t^\frown(i) \in T$ for each $i < 2$. *Sacks forcing* is the partial order consisting of perfect trees $T \subseteq 2^{<\omega}$, and a condition

S extends a condition T if S is a subtree of T. Sacks forcing is often viewed as a very minimally destructive way of forcing $\neg\mathsf{CH}$; the Sacks coding forcing defined below can be seen as a minimally destructive way of forcing a Δ^1_3 wellorder in the presence of large continuum. Throughout this section we assume $V = L[Y]$, where $Y \subseteq \omega_1$ is generic over L for a cardinal preserving forcing notion.

Definition 3.10. Fix $n^* \in \omega$. By induction on $i < \omega_1$, define a sequence $\overline{\mu} = \langle \mu_i : i \in \omega_1 \rangle$ such that μ_i is an ordinal and $(|\mu_i| = \aleph_0)^L$, for each $i < \omega_1$. Let $\mu_0 = \emptyset$, and supposing $\langle \mu_j : j < i \rangle$ has been defined, let μ_i be the least ordinal $\mu > \sup_{j<i} \mu_j$ such that :

(1) $L_\mu[Y \cap i] \prec_{\Sigma^1_{n^*}} L_{\omega_1}[Y \cap i]$;
(2) $L_\mu[Y \cap i] \models \mathsf{ZF}^-$;
(3) $L_\mu[Y \cap i] \models$ "ω is the largest cardinal".

Let $\mathcal{B}_i := L_{\mu_i}[Y \cap i]$. For $r \in 2^\omega$, we say that r *codes* Y *below* i if for all $j < i$, $j \in Y \Leftrightarrow \underbrace{L_{\mu_j}[Y \cap j][r]}_{=\mathcal{B}_j[r]} \models \mathsf{ZF}^-$. For a tree $T \subseteq 2^{<\omega}$, let $|T|$ denote the least $i < \omega_1$ such that $T \in \mathcal{B}_i$.

Definition 3.11. ([FF10, Definition 2]) *Sacks coding* is the partial order $C(Y)$ consisting of perfect trees $T \subseteq 2^{<\omega}$ such that r codes Y below $|T|$ whenever r is a branch through T. For $T_0, T_1 \in C(Y)$, let $T_1 \leq T_0$ if and only if T_1 is a subtree of T_0.

Remark 3.12. The hidden parameter n^* above is needed for the preservation results pertaining to definable combinatorial objects, e.g. item (5) of Fact 3.14 below. Specifically, n^* is chosen to an upper bound on the complexity of the formula expressing all relevant combinatorial properties which we want to reflect down to the models $\mathcal{B}_i$. In the present case of preserving a Π^1_1 tight mad family, $n^* = 5$ is sufficient. See also [BFB22, Remark 1].

Remark 3.13. Let G be $C(Y)$-generic over $L[Y]$. Since the definition of $C(Y)$ is absolute, it still holds in $L[Y][G]$ that if T is any condition in $C(Y)$ and r is a branch through T, in $L[Y][G]$,

$$Y \cap |T| = \{j < |T| \mid \mathcal{B}_j[r] \models \mathsf{ZF}^-\}$$

In particular the above holds for all $j < \sup\{|T| \mid T \in G\}$ whenever $r \in \bigcap G$.

Fact 3.14. The following hold.

(1) ([FF10, Lemma 4]) For any $j \in \omega_1$ and any $T \in C(Y)$ with $|T| \leq j$, there exists $T' \leq T$ such that $|T'| = j$.
(2) ([FF10, Lemma 7]) $C(Y)$ is proper.
(3) ([FF10, Lemma 6]) Let G be $C(Y)$-generic over V and let $r := \bigcap G$. Then in $V[G]$, r codes the set Y in the sense that for all $j < \omega_1$, $j \in Y \Leftrightarrow \mathcal{B}_j[r] \models \mathsf{ZF}^-$.
(4) ([FF10, Lemma 8]) $C(Y)$ is ω^ω-bounding.
(5) ([BFB22, Lemma 4.3]) Countable support iterations of $C(Y)$ preserve the tightness of Π^1_1 tight mad families which provably consist of constructible reals.

3.3. **A Δ^1_3 long wellorder.** With the ingredients provided by the previous sections we proceed with the proof of Theorem 3.1. Recall that $\diamondsuit$ is the assertion that there exists a sequence $\vec{A} = \langle A_\alpha : \xi < \omega_1\rangle$ where $A_\xi \subseteq \xi$ for each $\xi < \omega_1$ and for any $X \subseteq \omega_1$, the set $\{\xi < \omega_1 \mid X \cap \xi = A_\xi\}$ is stationary. $\vec{A}$ is called a $\diamondsuit$-sequence, and such a sequence can be constructed in L so that the sequence is Σ_1-definable over L_{ω_1}; see, for example, [Dev17, Theorem 3.3].

Proposition 3.15 ([FF10, Lemma 14])**.** Assume $\diamondsuit$ holds. Then there exists a sequence $\vec{S} = \langle S_\alpha : \alpha < \omega_2\rangle$ which is Σ_1-definable over L_{ω_2} consisting of sets $S_\alpha \subseteq \omega_1$ that are stationary costationary in L, and are almost disjoint in the sense that $|S_\alpha \cap S_\beta| < \omega_1$ for all distinct $\alpha, \beta < \omega_2$. Furthermore there exists a stationary $S_{-1} \subseteq \omega_1$ such that $S_{-1} \cap S = \emptyset$ for all $S \in \vec{S}$. Moreover, if $\mathcal{M}$, $\mathcal{N}$ are suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$, then $\vec{S}^{\mathcal{M}}$ and $\vec{S}^{\mathcal{N}}$ coincide on $\omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$. If $\mathcal{M}$ is suitable with $\omega_1^{\mathcal{M}} = \omega_1$, then $\vec{S}^{\mathcal{M}} = \vec{S} \restriction \omega_2^{\mathcal{M}}$.

We assume $V = L$, and therefore fix $\vec{S}$ and S_{-1} as above.

Lemma 3.16 ([FF10, Lemma 14])**.** There exists $F\colon \omega_2 \to L_{\omega_2}$ such that for all $a \in L_{\omega_2}$, $F^{-1}(a)$ is unbounded in ω_2, and F is Σ_1-definable over L_{ω_2}. Moreover, if $\mathcal{M}$, $\mathcal{N}$ are suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$, then $F^{\mathcal{M}}$ and $F^{\mathcal{N}}$ coincide on $\omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$. If $\mathcal{M}$ is suitable with $\omega_1^{\mathcal{M}} = \omega_1$, then $F^{\mathcal{M}} = F \restriction \omega_2^{\mathcal{M}}$.

Fix F as above. By Fact 3.14, item (5), tight mad families are indestructible by countable support iterations of $C(Y)$ when they satisfy an extra definability assumption; such a mad family can be constructed in L by the following.

Lemma 3.17 ([BFB22, Lemma 4.2])**.** If $V = L$ then there is a Π^1_1 tight mad family $\mathcal{A}$ such that ZFC proves $\mathcal{A}$ is a subset of L.

By recursion on $\alpha < \omega_2$, define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$, where $\mathbb{P}_0$ is taken to be the trivial poset. Suppose $\mathbb{P}_\alpha$ has been defined, and let G_α be $\mathbb{P}_\alpha$-generic over L. The wellorder $<_\alpha$ on $L[G_\alpha]$ has a natural definition using the global wellorder $<_L$ of the universe L and the collection of $\mathbb{P}_\alpha$-names for reals. We can assume any name for a real $\dot{x}$ is *nice*, in that it is determined by countably many maximal antichains $A_n \subseteq \mathbb{P}_\alpha$ consisting of conditions deciding $\dot{x}(n)$. This allows us to suppose that if $\alpha < \beta < \omega_2$ and $\dot{x}$ is a $\mathbb{P}_\beta$-name which is not a $\mathbb{P}_\alpha$-name, then all $\mathbb{P}_\alpha$-names precede $\dot{x}$ with respect to $<_L$, as it takes longer to construct x. Whenever $x \in L[G_\alpha]$ is a real, let γ_x denote the minimal γ such that x has a nice $\mathbb{P}_\gamma$-name, and define σ_x^α to be the L-minimal nice $\mathbb{P}_{\gamma_x}$-name for x. In this way we can understand the reals of $L[G_\alpha]$ by considering the set $N = \{\sigma_x^\alpha \mid x \in L[G_\alpha] \cap \omega^\omega\}$, which is canonically wellordered by $<_L$; therefore for reals $x, y \in L[G_\alpha]$, let $x <_\alpha y$ if and only if $\sigma_x^\alpha <_L \sigma_y^\alpha$. Since $\sigma_x^\alpha = \sigma_x^\beta = \sigma_x^{\gamma_x}$ for any $\beta < \alpha$, $<_\beta$ is an initial segment of $<_\alpha$. Let $\dot{<}_\alpha$ denote a $\mathbb{P}_\alpha$-name for $<_\alpha$.

Fix a recursive coding $\cdot * \cdot : \omega^\omega \times \omega^\omega \to \omega^\omega$ by letting $x * y = \{2n \mid n \in x\} \cup \{2n+1 \mid n \in y\}$, and for a real x define $\Delta(x) := x * (\omega \setminus x)$. Lastly we fix an absolute way of coding the ordinals of ω_2^L.

Definition 3.18. Let $\beta < \omega_2^L$ and $X \subseteq \omega_1^L$. We say X is a *sufficiently absolute code for* β if there exists a formula $\psi(x, y)$ such that for any suitable model $\mathcal{M}$ containing $X \cap \omega_1^{\mathcal{M}}$, there exists a unique $\overline{\beta} \in \omega_2^{\mathcal{M}}$ such that $\psi(\overline{\beta}, X \cap \omega_1^{\mathcal{M}})$ holds in $\mathcal{M}$, and $\overline{\beta} = \beta$

in the case $\omega_1^{\mathcal{M}} = \omega_1^L$. Moreover, if $\mathcal{M}, \mathcal{N}$ are any suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$ and $X \cap \omega_1^{\mathcal{M}} \in \mathcal{M} \cap \mathcal{N}$, then it is the same $\overline{\beta} \in \omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$ such that $(\psi(\overline{\beta}, X \cap \omega_1^{\mathcal{M}}))^{\mathcal{M}}$ and $(\psi(\overline{\beta}, X \cap \omega_1^{\mathcal{N}}))^{\mathcal{N}}$.

Fact 3.19. ([FZ10, Fact 5]) Sufficiently absolute codes exist for every $\beta < \omega_2^L$.

Henceforth we fix the formula $\psi(x, y)$ above. Working in $L[G_\alpha]$, let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}_\alpha^0 * \dot{\mathbb{Q}}_\alpha^1$ be a $\mathbb{P}_\alpha$-name for a two-step iteration in which $\dot{\mathbb{Q}}_\alpha^0$ is a $\mathbb{P}_\alpha$-name for $\mathbb{Q}$ (Definition 2.12), and $\dot{\mathbb{Q}}_\alpha^1$ is a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha^0$-name for the trivial forcing, unless the following occurs: $F(\alpha) = \{\sigma_x^\alpha, \sigma_y^\alpha\}$ for some reals $x, y \in L[G_\alpha]$ such that $x <_\alpha y$. In this case set $x_\alpha = x$, $y_\alpha = y$ and define $\dot{\mathbb{Q}}_\alpha^1$ to be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha^0$-name for a three-step iteration $\dot{\mathbb{K}}_\alpha^0 * \dot{\mathbb{K}}_\alpha^1 * \dot{\mathbb{K}}_\alpha^2$, where:

(1) $\dot{\mathbb{K}}_\alpha^0$ is a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha^0$-name for the countable support iteration $\langle \mathbb{P}^0_{\alpha,\beta}, \mathbb{K}^0_{\alpha,n} : \beta \leq \omega, n \in \omega \rangle$, where $\mathbb{K}^0_{\alpha,m}$ is a $\mathbb{P}^0_{\alpha,m}$-name for $Q(S_{\alpha+m})$ for all $m \in \Delta(x_\alpha * y_\alpha)$.

(2) Let R_α be $\mathbb{Q}_\alpha^0$-generic over $L[G_\alpha]$, and let H_α be $\mathbb{K}_\alpha^0$-generic over $L[G_\alpha * R_\alpha]$. In $L[G_\alpha * R_\alpha * H_\alpha]$ fix:

- Subsets $W_\alpha, W_\eta \subseteq \omega_1$ such that W_α is a sufficiently absolute code for the ordinal α and W_η is a sufficiently absolute code for an ordinal η such that $L_\eta \models |\alpha| \leq \omega_1$;
- A real $x_\alpha \oplus y_\alpha$ recursively coding the pair (x_α, y_α);
- A subset $Z_\alpha \subseteq \omega_1$ coding $G_\alpha * R_\alpha * H_\alpha$.

Fix a computable bijection $\langle \cdot, \cdot, \cdot, \cdot \rangle : \mathcal{P}(\omega_1) \to \mathcal{P}(\omega_1)$, and for $X \subseteq \omega_1$ and $i < 4$ write $(X)_i$ for those elements of $\mathcal{P}(\omega_1)$ such that $X = \langle (X)_i : i < 4 \rangle$. Let $X_\alpha = \langle W_\alpha, x_\alpha \oplus y_\alpha, W_\eta, Z_\alpha \rangle$ and let $\varphi_\alpha = \varphi_\alpha(\omega_1, X_\alpha)$ be a sentence with parameters ω_1 and X_α such that φ_α holds if and only if: There exists an ordinal $\overline{\alpha} \in \omega_2$ such that $(X_\alpha)_0 = \overline{\alpha}$ and there exists a pair (x, y) such that $(X_\alpha)_2 = x \oplus y$ and for all $m \in \Delta(x * y)$, $S_{\overline{\alpha}+m}$ is nonstationary. Next, define $\dot{\mathbb{K}}_\alpha^1$ to be a $(\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha^0 * \mathbb{K}_\alpha^0)$-name for $\mathcal{L}(\varphi_\alpha)$.

(3) Let Y_α be $\mathbb{K}_\alpha^1$-generic over $L[G_\alpha * R_\alpha * H_\alpha]$. Then as $\{n \in \omega \mid 2n \in Y_\alpha\} = X_\alpha$ and X_α codes $G_\alpha * R_\alpha * H_\alpha$, we have $L[G_\alpha * R_\alpha * H_\alpha *$

$Y_\alpha] = L[Y_\alpha]$. In this model, let $\mathbb{K}^2_\alpha$ be a $(\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{K}}^0_\alpha * \dot{\mathbb{K}}^1_\alpha)$-name for $C(Y_\alpha)$.

This completes the definition of $\mathbb{P} = \mathbb{P}_{\omega_2}$. Then $\mathbb{P}$ is a countable support iteration such that each $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha$ is an S_{-1}-proper forcing notion of size $\leq \omega_1$, and by Theorem 2.22 and Fact 3.14, $\mathbb{Q}_\alpha$ is forced to strongly preserves the tightness of $\mathcal{A}$. Therefore:

Lemma 3.20. $\mathbb{P}$ is S_{-1}-proper, strongly preserves the tightness of $\mathcal{A}$, and has the $\aleph_2$-cc.

Proof. By Lemma 3.6, Lemma 2.3, and Lemma 1. □

Let G be $\mathbb{P}$-generic over L and $<_G = \bigcup <_\alpha$, where $<_\alpha = \dot{<}^G_\alpha$.

Lemma 3.21 ([FF10])**.** Let G be $\mathbb{P}$-generic over L, and let x, y be reals in $L[G]$. Then $x <_G y$ if and only if: $(*)$ there exists a real r such that for every countable suitable $\mathcal{M}$ containing r as an element, there exists $\overline{\alpha} < \omega_2^{\mathcal{M}}$ such that for all $m \in \Delta(x * y)$, $S^{\mathcal{M}}_{\overline{\alpha}+m}$ is nonstationary in $\mathcal{M}$. Hence, $<_G$ is a Δ^1_3 definable wellorder of the reals in $L[G]$.

The complexity of $<_G$ is explicitly calculated as follows. We have $x <_G y \Leftrightarrow \Phi(x, y)$, where $\Phi(x, y)$ is the formula

$$\underbrace{\exists r \in \mathcal{P}(\omega^\omega)}_{\exists}[\underbrace{\forall \mathcal{M} \text{ countable, suitable, } r \in \mathcal{M}}_{\forall}$$
$$\underbrace{(\exists \alpha < \omega_2^{\mathcal{M}}}_{\exists}\ \underbrace{(\mathcal{M} \models \forall m \in \Delta(x * y)\ S_{\alpha+m} \in \mathrm{NS}_{\omega_1}))}_{\Delta^1_1}]$$

Thus, $\Phi(x, y)$ is a Σ^1_3 formula. However, $<_G$ is also a total wellorder, since if x, y are any reals in $L[G]$, there exists $\alpha = \max(\gamma_x, \gamma_y) < \omega_2$ such that $x = (\sigma^\alpha_x)^G$ and $y = (\sigma^\alpha_y)^G$. Either $\sigma^\alpha_x <_L \sigma^\alpha_y$ or $\sigma^\alpha_y <_L \sigma^\alpha_x$; in the case $\neg(x <_G y)$ then we must have $y <_G x$. Therefore the complement of $<_G$ is Σ^1_3-definable, giving that $<_G$ is Δ^1_3-definable.

Proof of Theorem 3.1. Suppose $V = L$, and fix a Π^1_1 definable tight mad family $\mathcal{A}$ from Lemma 3.17. Let $\mathbb{P}$ be the ω_2-length countable support iteration constructed in this section, and let G be $\mathbb{P}$-generic over V. By Lemma 3.20, $\mathbb{P}$ is S_{-1}-proper. Define $<_G = \bigcup_{\alpha<\omega_2} \dot{<}_\alpha^G$; by Lemma 3.21, $<_G$ is a Δ^1_3 wellorder of the reals. Since any set $\mathcal{S} \subseteq [\omega]^\omega \cap V[G]$ such that $|\mathcal{S}| < \omega_2$ appears at some inital stage $\alpha < \omega_2$ of the iteration, by definition $\dot{\mathbb{Q}}^0_\alpha$ is a $\mathbb{P}_\alpha$-name for the forcing $\mathbb{Q}$ of Definition 2.12, so $\mathcal{S}$ is not splitting in $V[G_{\alpha+1}]$. Therefore $(\mathfrak{s} = \aleph_2)^{V[G]}$. Next notice that by Shoenfield absoluteness, $\mathcal{A}$ is defined by the same Π^1_1 formula in $V[G]$ as in V. Moreover $\mathcal{A}$ remains a tight mad family in $V[G]$ by Lemma 3.20, and therefore $\mathcal{A}$ provides a witness for $(\mathfrak{a} = \aleph_1)^{V[G]}$. This completes the proof. □

4. Definable Spectra

In this section we consider projective mad families, with a definition of optimal complexity, which are of size $\kappa > \aleph_1$. We briefly give an account of the work in this direction thus far.

Friedman and Zdomskyy [FZ10] established that a tight mad family with optimal projective definition is consistent with $\mathfrak{b} = \mathfrak{c} = \aleph_2$ and this was extended by Fischer, Friedman and Zdomskyy [FFZ11] to $\mathfrak{b} = \mathfrak{c} = \aleph_3$. By [Rag09] no tight mad family can contain a perfect subset, and the Mansfield-Solovay theorem states that any Σ^1_2 set is either a subset of L, or contains a perfect set of nonconstructible reals. Therefore, no Σ^1_2 tight mad family can exist in a model of $\mathfrak{b} > \aleph_1$; the optimal definition of a tight mad family of size greater than $\aleph_2$ is thus Π^1_2. Dropping the tightness requirement, Brendle and Khomskii [BK13] showed $\mathfrak{b} = \mathfrak{c} \geq \aleph_2$, is consistent with a Π^1_1 mad family; one can moreover have a Δ^1_3 wellorder of the reals in such a model by Fischer, Friedman, and Khomskii [FFK13]. The first work on projective witnesses of size κ when $\aleph_1 < \kappa < \mathfrak{c}$ is done by Fischer, Friedman, Schrittesser, and Törnquist [FFST25]; again by the Mansfield-Solovay theorem, the best possible complexity of such an object is Π^1_2.

So far the attention has been on finding definable mad families witnessing the value of $\mathfrak{a}$, the minimal element of the set

$\mathrm{spec}(\mathfrak{a}) = \{|\mathcal{A}| \mid \mathcal{A} \subseteq [\omega]^\omega \text{ is mad}\}$. The study of this set was pioneered by Hechler [Hec72] and followed up by Blass [Bla93], and Shelah and Spinas [SS15]. Similar considerations have since been taken for the cardinals $\mathfrak{a}_T$ and $\mathfrak{i}$; see [FS25], [Bri24] for the former, and [FS19], [FS22] for the latter. Given that now there is fairly substantial knowledge of how to control which $\kappa \in \mathrm{spec}(\mathfrak{a})$ as well as the definability of mad families of size $\mathfrak{a} = \min(\mathrm{spec}(\mathfrak{a}))$ it is natural we try to combine these lines of inquiry, by asking that for every $\kappa \in \mathrm{spec}(\mathfrak{a})$ there is a projective mad family of size κ with an optimal definition. Here we obtain:

Theorem 4.1. *It is consistent with $\mathfrak{a} = \aleph_1 < \mathfrak{c} = \aleph_2$ that there exists a Π^1_1-definable tight mad family of size $\aleph_1$, and a Π^1_2-definable tight mad family of size $\aleph_2$.*

The strategy in proving the above statement will be as follows. We begin in a model of $V = L$ and fix a Π^1_1 tight mad family $\mathcal{A}_1$. We recursively define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$, along the way constructing a Π^1_2 tight mad family $\mathcal{A}_2$ consisting of ω_2-many $\mathbb{Q}_\alpha$-generic reals a_α, arising from a certain forcing notion $\mathbb{K}_\alpha$, and such that each $\mathbb{Q}_\alpha$ is forced by $\mathbb{P}_\alpha$ to be an S-proper poset strongly preserving the tightness of $\mathcal{A}_1$. The tightness of $\mathcal{A}_2$ is taken care of by appropriate bookeeping of those countable sequences of reals appearing in $\mathcal{I}(\mathcal{A}^2_\alpha)^+$, where $\mathcal{A}^2_\alpha$ denotes the part of $\mathcal{A}_2$ constructed up to stage α. The definition will be a slight modification of the iteration defined in the proof of [FZ10, Theorem 1]. The main modification is that, whereas [FZ10] added Hechler reals cofinally often along the iteration, so to increase $\mathfrak{b}$ and ensure there are no mad families of size $\aleph_1$, in the present construction, cofinally often we take $\mathbb{Q}_\alpha$ to be some S-proper forcing of size $\aleph_1$. $\mathbb{Q}_\alpha$ is reserved for future applications. The intention behind this use of Hechler forcing in [FZ10] was pointed out in [FZ10, Question 18], suggesting that it was unknown to the authors whether the iterand $\mathbb{K}_\alpha$ itself added a dominating real. Note that Proposition 4.10 below will show that this is not the case.

The Π^1_2 definability of $\mathcal{A}_2$ is achieved similarly to that of the Δ^1_3 wellorder of the previous section, however differs in subtle but important ways. Namely, using $\vec{S} = \langle S_\alpha : \alpha < \omega_2 \rangle$ and S_{-1}, as in Proposition 3.15, the $\mathbb{K}_\alpha$-generic real a_α will code a stationary/nonstationary pattern into the αth ω-block of $\vec{S}$ by coding a sequence of clubs $\vec{C}_\alpha = \langle C_{\alpha+m} : m \in \Delta(a_\alpha) \rangle$, which are also added by $\mathbb{K}_\alpha$. Therefore conditions will consist of a finite part, making approximations to a_α, and an infinite part, making countably many approximations to a club in $\omega_1 \setminus S_{\alpha+m}$, for m belonging to the finite part. This simultaneous construction must be carried out carefully to ensure $S_{\alpha+m}$ remains stationary when $m \notin \Delta(a_\alpha)$. The lightface projective definition is obtained again relying on the localization techniques appearing in the last section, so $\mathbb{K}_\alpha$ must also add $\langle Y_{\alpha+m} : m \in \Delta(a_\alpha) \rangle$ of sufficiently absolute codes for α. In the final extension $V[G]$, for all $a \in V[G] \cap [\omega]^\omega$ we have

$$\begin{aligned} a \in \mathcal{A}_2 \Leftrightarrow & \forall \mathcal{M} (\mathcal{M} \text{ is a countable suitable model and } a \in \mathcal{M}) \\ & \exists \overline{\alpha} < \omega_2^{\mathcal{M}} \ (\mathcal{M} \models \forall m \in \Delta(a) \ S_{\overline{\alpha}+m} \text{ is nonstationary}). \end{aligned}$$

The right hand side is in the form $\forall\exists$ and thus is a Π^1_2 formula.

For the proof of Theorem 4.1 we first establish preliminaries. Throughout the rest of the section we work in a model of $V = L$ unless explicitly stated otherwise. Fix a coanalytic tight mad family $\mathcal{A}_1$, as well as $\vec{S} = \langle S_\alpha : \alpha < \omega_2 \rangle$ and $S_{-1} \subseteq \omega_1$ given by Proposition 3.15. Let $F\colon \mathrm{Lim}(\omega_2) \to L_{\omega_2}$ be such that $F^{-1}(x)$ is unbounded in ω_2 for all $x \in L_{\omega_2}$. By [FF10, Lemma 14], we can take $\vec{S}$ and F to be Σ_1-definable over L_{ω_2}, and moreover that whenever $\mathcal{M}, \mathcal{N}$ are suitable models with $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$, then $\vec{S}^{\mathcal{M}}$ and $\vec{S}^{\mathcal{N}}$ agree on $\omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$. Finally, for all $\alpha < \omega_2^L$, let X_α denote a sufficiently absolute code for α. Whereas the coding of the Δ^1_3 wellorder was achieved by the $C(Y)$-generic reals, the generic reals in the current context will result from almost disjoint coding, a method developed by Solovay and Jensen [SJ70], and for which we give a general definition.

Definition 4.2. Let $\mathcal{R}$ be an almost disjoint family in V, and let $X \in V$ be a subset of ω_1. The *almost disjoint coding of* X *with respect to* $\mathcal{R}$ is the partial order $\mathbb{P}_{\mathcal{R}}(X)$ consisting of conditions (s, F) such that s is a finite subset of ω and F is a finite subset of $\{r_\xi \mid \xi \in X\} \subseteq \mathcal{R}$. The extension relation is defined by letting $(t, G) \leq (s, F)$ if and only if

(1) t end-extends s, $G \supseteq F$;
(2) For all $\xi \in X$ and $r_\xi \in F$, $(t \setminus s) \cap r_\xi = \emptyset$.

Fact 4.3. $\mathbb{P}_{\mathcal{R}}(X)$ is σ-centered. If G is $\mathbb{P}_{\mathcal{R}}(X)$-generic over V, let $a := \bigcup\{s \mid \exists F(s, F) \in G\}$. Then G and a are mutually definable in the generic extension, and in $V[G] = V[a]$, for all $\xi < \omega_1$, $\xi \in X$ if and only if $a \cap r_\xi$ is finite.

For proofs of the above, see, for example, [Har77] or [Jec97, Example IV]. Using [FZ10, Proposition 3] we fix an almost disjoint family $\mathcal{R} = \{R_{\langle \eta, \xi \rangle} \mid \eta \in \omega \cdot 2, \xi \in \omega_1\}$ which is Σ_1-definable over L_{ω_1}, and such that for every suitable model $\mathcal{M}$, $\mathcal{R} \cap \mathcal{M} = \{R_{\eta,\xi} \mid \eta \in \omega \cdot 2, \xi \in \omega_1^{\mathcal{M}}\}$. For technical reasons in the coding, we modify the definition of the function $\Delta \colon \omega^\omega \to \omega^\omega$ given in the previous section by letting, for $s \subseteq \omega$, finite or infinite, $\Delta(s) = \{2n+1 \mid n \in s\} \cup \{2n+2 \mid n \in (\sup s \setminus s)\}$. Let $C(s) = \Delta(s) \cup (\omega \setminus \max(\Delta(s)))$. We think of $C(s)$ as the "coding area" associated with s. Denote by $E(s), O(s)$ the sets $\{s(2n) \mid 2n < |s|\}$ and $\{s(2n+1) \mid 2n+1 < |s|\}$ respectively. For a limit ordinal γ and a function $r \colon \gamma \to 2$, let $\mathrm{Even}(r) = \{\alpha < \gamma \mid r(2\alpha) = 1\}$ and $\mathrm{Odd}(r) = \{\alpha < \gamma \mid r(2\alpha+1) = 1\}$. Lastly, for ordinals $\alpha < \beta$, let $\beta - \alpha$ denote the ordinal γ such that $\alpha + \gamma = \beta$. If B is a set of ordinals, let $B - \alpha = \{\beta - \alpha \mid \beta \in \beta\}$. If δ is an indecomposable ordinal then $(\alpha + B) \cap \delta - \alpha = B \cap \delta$.

Continuing with the definition of $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ and the construction of a Π^1_2 tight mad family $\mathcal{A}_2 = \{a_\alpha \mid \alpha < \omega_2\}$, suppose $\alpha < \omega_2$ and $\mathbb{P}_\alpha$ has been defined. Let G_α be a $\mathbb{P}_\alpha$-generic filter, and let $\dot{\mathcal{A}}^2_\alpha$ be a $\mathbb{P}_\alpha$-name for the set of elements of $\mathcal{A}_2$ constructed up to stage α. The density arguments of Lemma

4.11 require the following inductive assumption:

$$(*) \quad \forall r \in \mathcal{R} \forall A' \in [\mathcal{A}^2_\alpha]^{<\omega} \ (|\mathrm{E}(r) \setminus \bigcup A'| = |\mathrm{O}(r) \setminus \bigcup A'| = \aleph_0),$$

Since $\mathcal{R}$ is an almost disjoint family, $(*)$ implies that for every $A' \in [\mathcal{A}^2_\alpha \cup \mathcal{R}]^{<\omega}$ and $r \in \mathcal{R} \setminus A'$, also $|E(r) \setminus \bigcup A'| = |O(r) \setminus \bigcup A'| = \aleph_0$, as otherwise $r \cap r'$ is infinite for some $r' \in A' \cap \mathcal{R}$.

If $\alpha < \omega_2$ is a successor ordinal, let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}^0_\alpha$ be a $\mathbb{P}_\alpha$-name for a proper poset of cardinality $\aleph_1$ such that $\Vdash_{\mathbb{P}_\alpha}$"$\dot{\mathbb{Q}}^0_\alpha$ strongly preserves the tightness of $\mathcal{A}_1$". For limit $\alpha \in \omega_2$, unless explicitly mentioned otherwise, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the trivial poset. Suppose $F(\alpha)$ is a sequence $\langle \dot{x}_i : i \in \omega \rangle$ of $\mathbb{P}_\alpha$-names such that $x_i = \dot{x}_i^{G_\alpha}$ is an infinite subset of ω such that for all $i \in \omega$, $x_i \in \mathcal{I}(\mathcal{A}^2_\alpha)^+$. The assumption $(*)$ and the almost disjointness of $\mathcal{R}$ imply the existence of a limit ordinal $\eta_\alpha \in \omega_1$ with the property that there exist no finite subsets J, E of $\omega \cdot 2 \times (\omega_1 \setminus \eta_\alpha), \mathcal{A}^2_\alpha$, respectively, and $i \in \omega$ such that $x_i \subseteq \bigcup_{\langle \eta, \xi \rangle \in J} R_{\langle \eta, \xi \rangle} \cup \bigcup E$. Fix such an $\eta_\alpha \in \omega_1$ as well as $Z_\alpha \subset \omega$ coding a surjection of ω onto η_α. The following is the original definition from [FZ10, Section 3].

Definition 4.4. (Friedman, Zdomskyy; [FZ10]) The partial order $\mathbb{K}_\alpha$ consists of all pairs $p = \langle \langle s, s^* \rangle, \langle c_k, y_k : k \in \omega \rangle \rangle$, such that:

(1) $c_k \subseteq \omega_1 \setminus \eta_\alpha$ is closed bounded such that $S_{\alpha+k} \cap c_k = \emptyset$;
(2) $y_k \colon |y_k| \to 2$ is a function, where $|y_k| \in \omega_1$ is a limit, such that
   - $|y_k| > \eta_\alpha$, $y_k \restriction \eta_\alpha = 0$;
   - for all $\gamma < |y_k|$, $y_k(\eta_\alpha + 2\gamma) = 1$ if and only if $\gamma = 0$ or $\gamma \in X_\alpha$, so that $\mathrm{Even}(y_k) = (\{\eta_\alpha\} \cup (\eta_\alpha + X_\alpha)) \cap |y_k|$.[1]
(3) $s \in [\omega]^{<\omega}$ and s^* is a finite subset of the set $\{R_{\langle m, \xi \rangle} \mid m \in \Delta(s), \xi \in c_m\} \cup \{R_{\langle \omega+m, \xi \rangle} \mid m \in \Delta(s),\ y_m(\xi) = 1\} \cup \mathcal{A}^2_\alpha$. Additionally, for all $n \in \omega$ such that $2n < |s \cap R_{\langle 0,0 \rangle}|$, $n \in Z_\alpha$ if and only if there exists $m \in \omega$ such that $(s \cap R_{\langle 0,0 \rangle})(2n) = R_{\langle 0,0 \rangle}(2m)$;
(4) For all $k \in C(s)$ and all limit $\gamma \in \omega_1$ such that $\eta_\alpha < \gamma \leq |y_k|$, if γ is a limit point of c_k and $\gamma = \omega_1^{\mathcal{M}}$ for some countable suitable model $\mathcal{M}$ containing both $y_k \restriction \gamma$ and $c_k \cap \gamma$ as elements, then

[1] Recall X_α denotes the sufficiently absolute code for α given by Fact 3.19.

the following holds in $\mathcal{M}$: "$[\mathrm{Even}(y_k) - \min(\mathrm{Even}(y_k))] \cap \gamma$ is the code of some limit ordinal $\overline{\alpha} \in \omega_2$ such that $S_{\overline{\alpha}+k}$ is nonstationary."

For $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ and $q = \langle\langle t, t^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ in $\mathbb{K}_\alpha$, define q to extend p and write $q \leq p$ if and only if:

(1) t end-extends s, $t^* \supseteq s^*$, and for all $x \in s^*$, $(t \setminus s) \cap x = \emptyset$;
(2) For all $k \in C(t)$, d_k end-extends c_k and $z_k \supseteq y_k$.

For $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$, let $\mathrm{Fin}(p) = \langle s, s^*\rangle$ denote the finite part of p, and let $\mathrm{Inf}(p) = \langle c_k, y_k : k \in \omega\rangle$ denote the infinite part of p. When $p, q \in \mathbb{K}_\alpha$ and $q \leq p$, we say q is a *pure extension of* p if $\mathrm{Fin}(p) = \mathrm{Fin}(q)$.

Proposition 4.10 will give the properness of $\mathbb{K}_\alpha$, as well as the preservation of $\mathcal{A}_1$. For the proof however, we will need some intermediary lemmas:

Lemma 4.5 ([FF10, Lemma 1])**.** For every $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ in $\mathbb{K}_\alpha$ and every $\gamma \in \omega_1$, there exists a pure extension $q \leq p$ with $\mathrm{Inf}(q) = \langle d_k, z_k : k \in \omega\rangle$, such that $|z_k| \geq \gamma$ and $\max(d_k) \geq \gamma$, for every $k \in \omega$.

The following notion appears implicitly in [FZ10].

Definition 4.6. For a condition $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ in $\mathbb{K}_\alpha$ and open dense $D \subseteq \mathbb{K}_\alpha$, we say p is *preprocessed* for D if and only if for every extension $q = \langle\langle t, t^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle \leq p$, if $q \in D$, then already there is some t_2^* such that $q' = \langle\langle t, t_2^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$ extending p, and $q' \in D$.

Note that if $p \in \mathbb{K}_\alpha$ is preprocessed for D and $r \leq p$, then r is preprocessed for D.

Lemma 4.7 ([FZ10, Claim 9])**.** For any $p \in \mathbb{K}_\alpha$ and open dense $D \subseteq \mathbb{K}_\alpha$, there exists a pure extension $q \leq p$ such that q is preprocessed for D.

Lemma 4.8. Let $q \in \mathbb{K}_\alpha \cap M$, where $M \prec H_\theta$ is a countable elementary submodel containing $\mathbb{K}_\alpha$ and $\mathcal{A}_1$, and let $\dot{Z} \in M$ be a

$\mathbb{K}_\alpha$-name for an element of $\mathcal{I}(\mathcal{A}_1)^+$. Then

$$W = \{m \in \omega \mid \exists p \leq q \ (\mathrm{Fin}(p) = \mathrm{Fin}(q) \wedge p \Vdash m \ \in \dot{Z})\}$$

is an element of $\mathcal{I}(\mathcal{A}_1)^+ \cap M$.

Proof. Fix a finite $F \subseteq \mathcal{A}_1 \cap M$. By assumption on $\dot{Z}$ we have that $q \Vdash_{\mathbb{K}_\alpha} \dot{Z} \setminus \bigcup F \in [\omega]^\omega$.

Lemma 4.9. For all $q \in \mathbb{K}_\alpha$ and $\dot{X}$ a $\mathbb{K}_\alpha$-name for an infinite subset of ω, there exists $p \leq q$ with $\mathrm{Fin}(p) = \mathrm{Fin}(q)$ and there exists $m^p \in \omega$ such that $p \Vdash m^p = \dot{X}(j)$.

Proof. Fix $q = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$, $\dot{X}$ as above. Consider the countable support product $\mathbb{P}_s := \prod_{k \in C(s)} Q^{\eta_\alpha}(S_{\alpha+k}) \times \mathcal{L}^{\eta_\alpha}(Y_{\alpha+k})$, where $Q^{\eta_\alpha}(S_{\alpha+k})$ consists of closed bounded subsets $c_k \subseteq \omega_1 \setminus \eta_\alpha$ such that $c_k \cap S_{\alpha+k} = \emptyset$ and is ordered by end extension; the partial order $\mathcal{L}^{\eta_\alpha}(Y_{\alpha+k})$ consists of functions $y_k \colon |y_k| \to 2$ with domain $|y_k|$, such that

- $|y_k| \in \omega_1 \setminus \eta_\alpha$ is a countable limit ordinal and $y_k \upharpoonright \eta_\alpha = 0$;
- $\mathrm{Even}(y_k) = (\{\eta_\alpha\} \cup (\eta_\alpha + X_\alpha)) \cap |y_k|$;
- for all $\gamma \leq |y_k|$, if $\gamma = \omega_1^{\mathcal{M}}$ for a suitable $\mathcal{M}$ with $y_k \upharpoonright \gamma \in \mathcal{M}$ and γ is a limit point of c_k, then $\mathcal{M} \models$"$\mathrm{Even}(y_k)$ is the code for some $\overline{\alpha} \in \omega_1$ such that $S_{\overline{\alpha}+k}$ is nonstationary ".

$\mathcal{L}^{\eta_\alpha}(Y_{\alpha+k})$ is ordered by end-extension. For notational simplicity we will suppress the superscript η_α in what follows. The ordering on $\mathbb{P}_s$ is given by $\langle d_k, z_k : k \in \omega\rangle \leq \langle c_k, y_k : k \in \omega\rangle$ if and only if d_k is an end-extension of c_k and $z_k \supseteq y_k$, i.e., if and only if $(d_k, z_k) \leq_{Q(S_{\alpha+k}) \times \mathcal{L}(Y_{\alpha+k})} (c_k, y_k)$. Here, $\leq_{Q(S_{\alpha+k})}$ denotes the ordering of the club shooting forcing of Definition 3.2, however with the modification that condition are closed bounded subsets containing only ordinals strictly greater than η_α. Likewise, $\leq_{\mathcal{L}(Y_{\alpha+k})}$ denotes the extension relation for the localization forcing of Definition 3.8, where the Σ_1-formula being localized is $\varphi(\omega_1, X_\alpha)$ which asserts "$\mathrm{Even}(y_k) - \min(\mathrm{Even}(y_k))$ is the code for some $\overline{\alpha} \in \omega_1$ such that $S_{\overline{\alpha}+k}$ is nonstationary". Note that if $\langle c_k, y_k : k \in \omega\rangle \Vdash_{\mathbb{P}_s}$

$\dot{X}(j) = m_j$, then $\langle\langle\emptyset,\emptyset\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \Vdash_{\mathbb{K}_\alpha} \dot{X}(j) = m_j$. Moreover $\mathbb{P}_s$ is a complete suborder of $\mathbb{P}_\emptyset$.

For all $k \in C(s)$, find c'_k, y'_k such that $c'_k \leq_{Q(S_{\alpha+k})} c_k$ and $y'_k \leq_{\mathcal{L}(Y_{\alpha+k})} y_k$, and $(c_k, y_k) \Vdash_{Q(S_{\alpha+k})\times\mathcal{L}(Y_{\alpha+k})} \dot{X}(j) = \check{m}_j$ for some $m_j \in \omega$. Then $\langle c'_k, y'_k : k \in \omega\rangle \in \mathbb{P}_s$ is an extension of $\langle c_k, y_k : k \in \omega\rangle$ and forces $\dot{X}(j) = \check{m}_j$. Therefore $\langle \mathrm{Fin}(q), \langle c'_k, y'_k : k \in \omega\rangle\rangle \leq_{\mathbb{K}_\alpha} q$ and decides $\dot{X}(j)$. □

Now, letting $\dot{X}$ be a $\mathbb{K}_\alpha$-name for $\dot{Z} \setminus \bigcup F$, for every $k \in \omega$ there exists $j > k$ and a pure extension $q_j \leq q$ in $\mathbb{K}_\alpha$, and there exists $m_j \in \omega$ with $q \Vdash \dot{X}(j) = \check{m}_j$. Then

$$Y_k = \{m_j \mid j > k, q_j \leq q \wedge q_j \Vdash \check{m}_j = \dot{X}(j)\}$$

is an infinite set witnessing $W \setminus \bigcup F$ is infinite. □

Proposition 4.10. For every $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$, every θ sufficiently large and countable elementary submodel $M \prec H_\theta$ containing $p, \mathbb{K}_\alpha, \mathcal{A}_1$, and every $B \in \mathcal{I}(\mathcal{A}_1)$ such that $B \cap Y$ is infinite for all $Y \in \mathcal{I}(\mathcal{A}_1)^+ \cap M$, if $M \cap \omega_1 = j \notin \bigcup_{k\in C(s)} S_{\alpha+k}$, then there is an $(M, \mathbb{K}_\alpha, \mathcal{A}_1, B)$-generic condition $q \leq p$ such that $\mathrm{Fin}(q) = \mathrm{Fin}(p)$.

Proof. Let θ be a sufficiently large regular cardinal and let $M \prec H_\theta$ be a countable elementary submodel containing $p, \mathbb{K}_\alpha$ and $\mathcal{A}_1$, such that $j = M \cap \omega_1 \notin \bigcup_{k\in C(s)} S_{\alpha+k}$. Fix $B \in \mathcal{I}(\mathcal{A}_1)$ such that $B \cap Y$ is infinite for all $Y \in \mathcal{I}(\mathcal{A}_1)^+ \cap M$. Let $\{D_n \mid n \in \omega\}$ enumerate all open dense subsets of $\mathbb{K}_\alpha$ in M, and let $\{\dot{Z}_n \mid n \in \omega\}$ enumerate all $\mathbb{K}_\alpha$-names for subsets of ω in M which are forced to be in $\mathcal{I}(\mathcal{A}_1)^+$ such that each name appears infinitely often. Let $\langle j_n : n \in \omega\rangle$ be an increasing cofinal sequence of ordinals converging to j. We inductively define a descending sequence $\langle q_n : n \in \omega\rangle \subseteq M \cap \mathbb{K}_\alpha$ where $q_n = \langle\langle s, s^*\rangle, \langle d^n_k, c^n_k : k \in \omega\rangle\rangle$ and such that:

(1) $d^0_k = c_k$, $z^0_k = y_k$;
(2) For all $n \in \omega$ and $k \in C(s)$, d^{n+1}_k is an end-extension of d^n_k and $z^{n+1}_k \supseteq z^n_k$;

(3) $\max(d_k^{n+1}), |z_k^{n+1}| \geq j_n$;
(4) q_{n+1} is preprocessed for D_n;
(5) $q_{n+1} \Vdash (\dot{Z}_n \cap B) \setminus n \neq \emptyset$.

Assume q_n has been constructed. First extend q_n with a pure extension q_n' such that q_n is preprocessed for D_n. Next let

$$W_{n+1} = \{m \in \omega \mid \exists r \leq q_n'(\mathrm{Fin}(r) = \mathrm{Fin}(q) \wedge\ r \Vdash m \in \dot{Z}_n)\}.$$

Since $W_{n+1} \in \mathcal{I}(\mathcal{A}_1)^+$ by Lemma 4.8, fix $m_n \in \omega$ such that $m_n > n$ and $m_n \in W_{n+1} \cap B$. Let $r \leq q_n'$ be given by $m_n \in W$, and let $q_{n+1} \leq r$ be a pure extension of r such that $\max(d_k^{n+1}) \geq j_n$ and $|z_k^{n+1}| \geq j_n$. Then q_{n+1} satisfies the above clauses, so this completes the inductive construction.

Set $d_k = \bigcup_{n\in\omega} d_k^n \cup \{j\}$ and $z_k = \bigcup_{n\in\omega} z_k^n$ for all $k \in C(s)$, and for $k \notin C(s)$, let $d_k = z_k = \emptyset$. Define $q := \langle\langle s, s^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$. Then $q \in \mathbb{K}_\alpha$, which can be verified as in the proof of Lemma 4.7. It remains to see that q is an $(M, \mathbb{K}_\alpha, \mathcal{A}_1, B)$-generic condition.

First we show q is $(M, \mathbb{K}_\alpha)$-generic by showing that for all $n \in \omega$, $D_n \cap M$ is predense below q. Fix n and let $r = \langle\langle t, t^*\rangle, \langle d_k', z_k' : k \in \omega\rangle\rangle \leq q$, and we can assume $r \in D_n$. Then as $r \leq q_{n+1}$ and q_{n+1} is preprocessed for D_n, there is $r' = \langle\langle t, t_2^*\rangle, \langle d_k^{n+1}, z_k^{n+1} : k \in \omega\rangle \leq q_{n+1}$ for some finite $t_2^* \in M$, such that already $r' \in D_n$. Clearly $r' \in M$. Then r and r' are compatible, as witnessed by the condition $\langle\langle t, t^* \cup t_2^*\rangle, \langle d_k', z_k' : k \in \omega\rangle\rangle$.

Lastly, for all $n \in \omega$ we have $q \Vdash |\dot{Z}_n \cap B| = \omega$. Let $\ell \in \omega$, and take any $r \leq q$. Find $i > \ell$ such that $\dot{Z}_i = \dot{Z}_n$; then $r \leq q_{i+1}$ and so since q_{i+1} satisifes property (5), we have

$$r \Vdash \emptyset \neq (\dot{Z}_i \cap B) \setminus i = (\dot{Z}_n \cap B) \setminus i \subseteq (\dot{Z}_n \cap B) \setminus \ell.$$

As ℓ was arbitrary this shows q forces $\dot{Z}_n \cap B$ is infinite, and therefore q is an $(M, \mathbb{K}_\alpha, \mathcal{A}_1, B)$-generic condition. □

Let H_α be $\mathbb{K}_\alpha$-generic over $V[G_\alpha]$, and set $Y_k^\alpha = \bigcup_{p\in H_\alpha} y_k$, $C_k^\alpha = \bigcup_{p\in H_\alpha} c_k$, $a_\alpha = \bigcup_{p\in H_\alpha} s$, $\mathcal{A}_{\alpha+1}^2 = \mathcal{A}_2 \cup \{a_\alpha\}$. The following lemma gives consequences of forcing with $\mathbb{K}_\alpha$.

Lemma 4.11 ([FZ10, Claim 11])**.** The following hold.

(1) $a_\alpha \in [\omega]^\omega$ is almost disjoint from all elements of $\mathcal{A}^2_\alpha$;
(2) For all $i \in \omega$, $a_\alpha \cap x_i$ is infinite;
(3) For all $m \in \Delta(a_\alpha)$, C^α_m is a club in ω_1 such that $C^\alpha_m \cap S_{\alpha+m} = \emptyset$, and for all $\xi \in \omega_1$, $\xi \in C^\alpha_m$ if and only if $a_\alpha \cap R_{\langle m,\xi\rangle}$ is finite;
(4) For all $m \in \Delta(a_\alpha)$, $Y^\alpha_m \colon \omega_1 \to 2$ is a total function, and for all $\xi \in \omega_1$, $Y^\alpha_m(\xi) = 1$ if and only if $a_\alpha \cap R_{\langle \omega+m,\xi\rangle}$ is finite;
(5) For all $n \in \omega$, $n \in Z_\alpha$ if and only if there exists $m \in \omega$ such that $(a_\alpha \cap R_{\langle 0,0\rangle})(2n) = R_{\langle 0,0\rangle}(2m)$;
(6) For every $r \in \mathcal{R}$ and finite $A' \subseteq \mathcal{A}^2_{\alpha+1}$, $|E(r) \setminus \bigcup A'| = |O(r) \setminus \bigcup A'| = \omega$.

Lemma 4.12 ([FZ10, Corollary 12])**.** $\mathbb{K}_\alpha$ is S_{-1}-proper. Moreover, for every $p = \langle\langle s, s^*\rangle : \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$, the subposet $\mathbb{K}_\alpha \restriction p = \{r \in \mathbb{K}_\alpha \mid r \leq p\}$ is $(\omega_1 \setminus \bigcup_{n \in C(s)} S_{\alpha+n})$- proper.

This completes the definition of $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$.

Corollary 4.13. $\mathbb{P}_{\omega_2}$ is S_{-1}-proper and strongly preserves the tightness of $\mathcal{A}_1$. Moreover for all $m \in \omega \setminus \Delta(a_\alpha)$, $S_{\alpha+m}$ remains stationary in $L[G]$.

Proof. By Proposition 4.10, Lemmas 4.12 and 3.6. □

Lemma 4.14 ([FZ10, Lemma 13])**.** If G is $\mathbb{P}$-generic over L, then in $L[G]$, $\mathcal{A}_2$ is definable by the following Π^1_2 formula:

$$a \in \mathcal{A}_2 \Leftrightarrow \forall \mathcal{M}[(\mathcal{M} \text{ is a countable suitable model, } a \in \mathcal{M})$$
$$\exists \overline{\alpha} < \omega_2^{\mathcal{M}}(\mathcal{M} \models \forall m \in \Delta(a)(S_{\overline{\alpha}+m} \text{ is nonstationary})]$$

With this we are ready to prove the main theorem of this section:

Proof of Theorem 4.1 Let $\mathbb{P}$ be the countable support iteration defined above, let G be $\mathbb{P}$-generic over L, and let $\mathcal{A}_2 = \{a_\alpha \mid \alpha < \omega_2\}$, where $a_\alpha = \dot{a}^G_\alpha$ where $\dot{a}_\alpha$ is the generic real added by $\mathbb{Q}_\alpha$. By Lemma 4.11, item (1), $\mathcal{A}_2$ is an almost disjoint family of infinite subsets of ω. To see it is tight, suppose that there is $\{x_i \mid i \in \omega\} \in L[G]$ such that $x_i \in \mathcal{I}(\mathcal{A}_2)^+$ for every $i \in \omega$. Then there is $\alpha < \omega_2$ such that $\langle x_i : i \in \omega\rangle \in L[G_\alpha]$, where $G_\alpha = G \cap \mathbb{P}_\alpha$, so there is a sequence of $\mathbb{P}_\alpha$-names $\langle \dot{x}_i : i \in \omega\rangle \in L_{\omega_2}$ such that

$x_i = \dot{x}_i^{G_\alpha}$. Since $F^{-1}(\langle \dot{x}_i : i \in \omega \rangle)$ is unbounded in ω_2, there exists $\beta \geq \alpha$ such that $F(\beta) = \langle \dot{x}_i : i \in \omega \rangle$. By definition of $\dot{\mathbb{Q}}_\beta$, and Lemma 4.11 item (2), $a_\beta \cap x_i$ is infinite in $L[G_\beta]$ for all $i \in \omega$, where a_β is the $\mathbb{Q}_\beta$-generic real. As $a_\beta \in \mathcal{A}_2$, we have $(\mathcal{A}_2 \text{ is tight})^{L[G]}$. That $\mathcal{A}_2$ is Π^1_2-definable in $L[G]$ is by Lemma 4.14.

As $\mathcal{A}_1 \in L$ is Π^1_1-definable in L, by Shoenfield absoluteness $\mathcal{A}_1$ remains Π^1_1-definable in $L[G]$. By Proposition 4.10, for every $\alpha < \omega_2$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a proper forcing strongly preserving the tightness of $\mathcal{A}_1$. Thus $(\mathfrak{a} = |\mathcal{A}_1| = \aleph_1 < \mathfrak{c} = \aleph_2 = |\mathcal{A}_2|)^{L[G]}$. □

5. Constellations and Definability

With the above results, we can now prove our main theorem.

Theorem 5.1. *It is consistent that $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$, there exists a Δ^1_3 wellorder of the reals, as well as tight mad families of cardinality $\aleph_1$ and $\aleph_2$, which are respectively Π^1_1 and Π^1_2 definable.*

Proof. We work in a model of $V = L$. The following can be obtained analagously as in Lemma 3.15, using Solovay's theorem on the existence of ω_1-many pairwise disjoint stationary subsets of ω_1 and the relativized $\Diamond$ sequence $\Diamond_T$ for stationary $T \subseteq \omega_1$; the assertion $\Diamond_T$ holds under $V = L$ for any such T.

Lemma 5.2. There are pairwise disjoint stationary subsets $\{T_i\}_{i \in 3}$ of ω_1 such that for each $i \in 2$ there is a sequence $\vec{S}^i = \langle S^i_\alpha \subseteq T_i : \alpha < \omega_2 \rangle$ of stationary costationary subsets of ω_1, such that for distinct $\alpha, \beta < \omega_2$, $S^i_\alpha \cap S^i_\beta$ is bounded. Moreover, whenever $\mathcal{M}, \mathcal{N}$ are suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$, then $\langle (S^i_\alpha)^{\mathcal{M}} : \alpha < \omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}} \rangle = \langle (S^i_\alpha)^{\mathcal{N}} : \alpha < \omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}} \rangle$.

Let $\mathcal{A}_1$ be a coanalytic tight mad family, and fix Σ_1-definable bookeeping function $F\colon \mathrm{Lim} \cap \omega_2 \to L_{\omega_2}$, and a Σ_1-definable almost disjoint family $\mathcal{R}$ such that $F, \mathcal{R}$ are as in the proof of Corollary 3.1 and Definition 4.4. Define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$, with $\mathbb{P}_0$ being the trivial forcing. Let G_α be $\mathbb{P}_\alpha$-generic over L; the wellordering $<_\alpha$ on the reals of $L[G_\alpha]$ is defined exactly as for the proof of Theorem 3.1. Let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the two-step iteration $\dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{Q}}^1_\alpha$ such that $\dot{\mathbb{Q}}^0_\alpha$ is a $\mathbb{P}_\alpha$-name for $\mathbb{Q}$ of Definition 2.12. Unless one of the following cases occurs, $\dot{\mathbb{Q}}^1_\alpha$ is a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-name for the trivial forcing.

Case I: α is a limit ordinal and $F(\alpha) = \{\dot{b}_i \mid i \in \omega\}$ is a sequence of $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-names such that in $V[G_\alpha]$, $\dot{b}_i^G$ is an element of $\mathcal{I}(\mathcal{A}^2_\alpha)^+$ for each $i \in \omega$. Then let $\dot{\mathbb{Q}}^1_\alpha$ be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$ name for the forcing notion $\mathbb{K}_\alpha$ of Definition 4.4, with respect to the same countable limit ordinal $\eta_\alpha \in \omega_1$, and modifying item (1) by letting $c_k \subseteq \omega_1 \setminus \eta_\alpha$ be a closed bounded subset such that $S^0_{\alpha+k} \cap c_k = \emptyset$.

Case II: $F(\alpha) = \{\sigma^\alpha_x, \sigma^\alpha_y\}$ is a pair of $\mathbb{P}_\alpha$-names for reals in $L[G_\alpha]$ such that $\sigma^\alpha_x <_L \sigma^\alpha_y$ (i.e., $x = (\sigma^\alpha_x)^G <_\alpha y = (\sigma^\alpha_y)^G$). In this case define $\dot{\mathbb{Q}}^1_\alpha$ to be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-name for $\mathbb{Q}^1_\alpha = \mathbb{K}^0_\alpha * \mathbb{K}^1_\alpha * \mathbb{K}^2_\alpha$ as defined in the proof of Corollary 3.1, however modifying the definition of $\mathbb{K}^0_\alpha$ by taking closed bounded subsets of $\omega_1 \setminus S^1_{\alpha+k}$, for $k \in \Delta(x_\alpha * y_\alpha)$.

This completes the definition of $\mathbb{P} = \mathbb{P}_{\omega_2}$. Note that for all $\alpha < \omega_2$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for either a proper, a $T_1 \cup T_2$-proper, or a $T_0 \cup T_2$-proper forcing notion, and in each case $\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\mathcal{A}_1$. Therefore $\mathbb{P}$ is T_2-proper and so preserves ω_1 as well as the tightness of $\mathcal{A}_1$. Let G be $\mathbb{P}$-generic over L. For each $\alpha < \omega_2$, $\mathbb{Q}_\alpha = \mathbb{Q}$ adds a real not split by the ground model reals, so $(\mathfrak{s} = \aleph_2)^{L[G]}$. For cofinally many $\alpha < \omega_2$, $\mathbb{Q}_\alpha$ adds an infinite $a_\alpha \subseteq \omega$ such that $\mathcal{A}_2 = \{a_\alpha \mid \alpha < \omega_2\}$ is a tight mad family with a Π^1_2 definition in $L[G]$. Moreover the wellorder $<_G = \bigcup_{\alpha<\omega_2}$ can be shown to be a Δ^1_3 wellorder of the reals of $L[G]$, as in Theorem 3.1. Then altogether we have that $L[G]$ witnesses the conclusions of the theorem. $\square$

6. Concluding remarks and Questions

The constellation $\aleph_1 = \mathfrak{b} < \mathfrak{a} = \mathfrak{s}$, first established in [She84], is shown to be consistent with a Δ^1_3 wellorder of the reals in [FF10]. The consistency of $\mathfrak{a} = \mathfrak{c}$ with a Δ^1_3 wellorder of the reals and a tight projective witness of $\mathfrak{a}$ is established in [FZ10]. In our results $\mathfrak{a}$ stays small, with a definable witness. Of interest however remains:

Question 6.1. Is $\mathfrak{b} < \mathfrak{a}$ consistent with a Π^1_2 (tight) mad family?

A model of $\aleph_2 < \mathfrak{b} = \mathfrak{c}$ with a Π^1_1 witness to $\mathfrak{a}$ is due to Brendle and Khomskii [BK13]. This construction however relies heavily on the preservation of splitting families and thus of interest remains:

Question 6.2. Is it consistent with $\mathfrak{a} < \mathfrak{c}$ or even with $\mathfrak{a} < \mathfrak{s} = \mathfrak{c}$ that there exist coanalytic mad families of sizes $\mathfrak{a}$ and $\mathfrak{c}$?

Note that our results heavily depend on the use of countable support iterations and so our techniques can only yield models with $\mathfrak{c} = \aleph_2$. A natural question is:

Question 6.3. Is it consistent that $|\mathrm{spec}(\mathfrak{a})| \geq 3$ and for each $\kappa \in \mathrm{spec}(\mathfrak{a})$ there exists a projective mad family of size κ with an optimal definition?

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